

# LOCAL WELLPOSEDNESS OF CHERN-SIMONS-SCHRÖDINGER

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**ABSTRACT.** In this article we consider the initial value problem for the Chern-Simons-Schrödinger model in two space dimensions. This is a covariant NLS type problem which is  $L^2$  critical. For this equation we introduce a so-called heat gauge, and prove that, with respect to this gauge, the problem is locally well-posed for initial data which is small in  $H^s$ ,  $s > 0$ .

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## 1. INTRODUCTION

The two dimensional Chern-Simons-Schrödinger system is a nonrelativistic quantum model describing the dynamics of a large number of particles in the plane, which interact both directly and via a self-generated electromagnetic field. The variables we use to describe the dynamics are the scalar field  $\phi$  describing the particle system, and the electromagnetic potential  $A$ , which can be viewed as a one-form on  $\mathbb{R}^{2+1}$ . The associated covariant differentiation operators are defined in terms of the electromagnetic potential  $A$  as

$$D_\alpha := \partial_\alpha + iA_\alpha. \quad (1.1)$$

With this notation, the Lagrangian for this system is

$$L(A, \phi) = \frac{1}{2} \int_{\mathbb{R}^{2+1}} \text{Im}(\bar{\phi} D_t \phi) + |D_x \phi|^2 - \frac{g}{2} |\phi|^4 dx dt + \frac{1}{2} \int_{\mathbb{R}^{2+1}} A \wedge dA \quad (1.2)$$

Although the electromagnetic potential  $A$  appears explicitly in the Lagrangian, it is easy to see that locally  $L(A, \phi)$  only depends upon the electromagnetic field  $F = dA$ . Precisely, the Lagrangian is invariant with respect to the transformations

$$\phi \mapsto e^{-i\theta} \phi \quad A \mapsto A + d\theta \quad (1.3)$$

for compactly supported real-valued functions  $\theta(t, x)$ .

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Computing the Euler-Lagrange equations for the above Lagrangian, one obtains a covariant NLS equation for  $\phi$ , coupled with equations giving the electromagnetic field in terms of  $\phi$ , as follows:

$$\begin{cases} D_t \phi &= i D_\ell D_\ell \phi + ig |\phi|^2 \phi \\ \partial_t A_1 - \partial_1 A_t &= -J_2 \\ \partial_t A_2 - \partial_2 A_t &= J_1 \\ \partial_1 A_2 - \partial_2 A_1 &= -\frac{1}{2} |\phi|^2 \end{cases} \quad (1.4)$$

where we use  $J_i$  to denote

$$J_i := \text{Im}(\bar{\phi} D_i \phi)$$

Regarding indices, we use  $\alpha = 0$  for the time variable  $t$  and  $\alpha = 1, 2$  for the spatial variables  $x_1, x_2$ . When we wish to exclude the time variable in a certain expression, we switch from Greek indices to Roman. Repeated indices are assumed to be summed. We discuss initial conditions in §2.

The system (1.4) is a basic model of Chern-Simons dynamics [14, 5, 6, 13]. For further physical motivation for studying (1.4), see [15, 17, 22].

The system (1.4) has the gauge invariance (1.3). It is also Galilean-invariant and has conserved *charge*

$$M(\phi) := \int_{\mathbb{R}^2} |\phi|^2 dx$$

and *energy*

$$E(\phi) := \frac{1}{2} \int_{\mathbb{R}^2} |D_x \phi|^2 - \frac{g}{2} |\phi|^4 dx.$$

As the scaling symmetry

$$\phi(t, x) \rightarrow \lambda \phi(\lambda^2 t, \lambda x), \quad \phi_0(x) \rightarrow \lambda \phi_0(\lambda x); \quad \lambda > 0,$$

preserves the charge of the initial data  $M(\phi_0)$ ,  $L_x^2$  is the critical space for equation (1.4).

Local wellposedness in  $H^2$  is established in [2]. Also given are conditions ensuring finite-time blowup. With a regularization argument, [2] demonstrates global existence (but not uniqueness) in  $H^1$  for small  $L^2$  data.

Our goal in this paper is to establish local wellposedness for (1.4) in spaces over the full subcritical range  $H^s$  with  $s > 0$ . However, in order to state the result we need to first remove the gauge freedom by choosing a suitable gauge. This is done in the next section, which ends with our main result.

## 2. GAUGE SELECTION

In order to interpret the Chern-Simons-Schrödinger system (1.4) as a well-defined time evolution, we need to impose a suitable gauge condition which eliminates the gauge freedom described in (1.3).

One can relate the gauge fixing problem here to the similar difficulty occurring in the study of wave and Schrödinger maps. Both wave maps and Schrödinger maps are geometric evolution equations, and in such settings the function  $\phi$  takes values not in  $\mathbb{C}$ , but rather more generally in some (suitable) manifold  $M$ . A gauged system arises when considering evolution equations at the level of the (pullback of the) tangent bundle  $\phi^* TM$ , where  $\phi^*$  denotes the pullback.

A classical gauge choice is the *Coulomb gauge*, which is derived by imposing the constraint  $\nabla \cdot A_x = 0$ . In low dimension, however, the Coulomb gauge has unfavorable high  $\times$  high  $\rightarrow$  low interactions.

To overcome this difficulty in the  $d = 2$  setting of wave maps into hyperbolic space, Tao [19] introduced the caloric gauge as an alternative. See [21] for an application of the caloric gauge to large data wave maps in  $d = 2$  and [1] for an application to small data Schrödinger maps in  $d = 2$ . We refer the reader to [20, Chapter 6] for a lengthier discussion and a comparison of various gauges.

Unfortunately, the direct analogue of the caloric gauge for the Chern-Simons-Schrödinger system does not result in any improvement over the Coulomb gauge. Instead, in this article we adopt from [4] a different variation of the Coulomb gauge called the *parabolic gauge*. We shall also refer to it as the *heat gauge*. The defining condition of the heat gauge is

$$\nabla \cdot A_x = A_t \quad (2.1)$$

Differentiating in the  $x_1$  and  $x_2$  directions the second and third equations (respectively) in (1.4) yields

$$\begin{cases} \partial_t \partial_1 A_1 - \partial_1^2 A_t &= -\partial_1 \operatorname{Im}(\bar{\phi} D_2 \phi) \\ \partial_t \partial_2 A_2 - \partial_2^2 A_t &= \partial_2 \operatorname{Im}(\bar{\phi} D_1 \phi) \end{cases}$$

Adding these, we get

$$\partial_t (\nabla \cdot A_x) - \Delta A_t = -\partial_1 J_2 + \partial_2 J_1,$$

which, in view of the heat gauge condition (2.1), implies that  $A_t$  evolves according to the nonlinear heat equation

$$(\partial_t - \Delta) A_t = -\partial_1 J_2 + \partial_2 J_1 \quad (2.2)$$

Similarly, we obtain (coupled) parabolic evolution equations for  $A_1$  and  $A_2$ :

$$\begin{cases} (\partial_t - \Delta) A_1 &= -J_2 - \frac{1}{2} \partial_2 |\phi|^2 \\ (\partial_t - \Delta) A_2 &= J_1 + \frac{1}{2} \partial_1 |\phi|^2 \end{cases} \quad (2.3)$$

We still retain the freedom to impose initial conditions for the parabolic equations for  $A$  in (2.2) and (2.3), in any way that is consistent with the last equation in (1.4). We impose

$$A_t(0) = \nabla \cdot A_x(0) = 0$$

To see that such a choice is consistent with (1.4), observe that  $\nabla \cdot A_x(0) = 0$  coupled with the fourth equation of (1.4) yields the system

$$\begin{cases} \partial_1 A_1(t=0) + \partial_2 A_2(t=0) &= 0 \\ \partial_1 A_2(t=0) - \partial_2 A_1(t=0) &= -\frac{1}{2} |\phi_0|^2, \end{cases} \quad (2.4)$$

which in turn implies

$$\begin{cases} \Delta A_1(t=0) &= \frac{1}{2} \partial_2 |\phi_0|^2 \\ \Delta A_2(t=0) &= -\frac{1}{2} \partial_1 |\phi_0|^2 \end{cases} \quad (2.5)$$

Substituting (2.5) into (2.3) yields

$$\begin{cases} \partial_t A_1(t=0) &= -\operatorname{Im}(\bar{\phi} D_2 \phi) \\ \partial_t A_2(t=0) &= \operatorname{Im}(\bar{\phi} D_1 \phi), \end{cases}$$

which is exactly what we obtain directly from the second and third equations of (1.4) at  $t = 0$  with the choice  $A_t(t=0) \equiv 0$ .

So having imposed an additional equation in order to fix a gauge, we study the initial value problem for the system

$$\begin{cases} D_t \phi &= iD_\ell D_\ell \phi + ig|\phi|^2 \phi \\ \partial_t A_1 - \partial_1 A_t &= -\text{Im}(\bar{\phi} D_2 \phi) \\ \partial_t A_2 - \partial_2 A_t &= \text{Im}(\bar{\phi} D_1 \phi) \\ \partial_1 A_2 - \partial_2 A_1 &= -\frac{1}{2}|\phi|^2 \\ A_t &= \nabla \cdot A_x \end{cases} \quad (2.6)$$

with initial data

$$\begin{cases} \phi(0, x) &= \phi_0(x) \\ A_t(0, x) &= 0 \\ A_1(0, x) &= \frac{1}{2}\Delta^{-1}\partial_2|\phi_0|^2(x) \\ A_2(0, x) &= -\frac{1}{2}\Delta^{-1}\partial_1|\phi_0|^2(x) \end{cases} \quad (2.7)$$

Our main result is the following.

**Theorem 2.1.** *For any small initial data  $\phi_0 \in H^s(\mathbb{R}^2)$ ,  $s > 0$ , the equation (2.6) with initial data (2.7) has solution  $\phi(t, x) \in C([0, 1], H^s(\mathbb{R}^2))$ , which is the unique uniform limit of smooth solutions. In addition,  $\phi_0 \mapsto \phi$  is Lipschitz continuous from  $H^s(\mathbb{R}^2)$  to  $C([0, 1], H^s(\mathbb{R}^2))$ .*

We remark that ideally one would like to have global well-posedness for small data in  $L^2$ . Unfortunately, in our arguments we encounter logarithmic divergencies at nearly every step with respect to the  $L^2$  setting, making it impossible to achieve this goal.

Another interesting remark is that while the initial system (1.4) is time reversible, the parabolic evolutions added by our gauge choice remove the time reversibility. One may possibly view this as a disadvantage of our gauge choice.

Our result is proved via a fixed point argument in a topology  $X^s$ , defined later, which is stronger than the  $C([0, 1], H^s(\mathbb{R}^2))$  topology. Thus we directly obtain uniqueness in  $X^s$ , as well as Lipschitz dependence on the initial data with respect to the  $X^s$  topology.

### 3. REDUCTIONS USING THE HEAT GAUGE

Let  $\tilde{f}$  denote the space-time Fourier transform

$$\tilde{f}(\tau, \xi) := \iint e^{-i(t\tau + x \cdot \xi)} f(t, x) dt dx$$

We define  $H^{-1}$  as the Fourier multiplier

$$H^{-1}f := \frac{1}{(2\pi)^3} \int \frac{1}{i\tau + |\xi|^2} e^{i(t\tau + x \cdot \xi)} \tilde{f}(\tau, \xi) d\tau d\xi \quad (3.1)$$

Applied to initial data, it takes the form

$$H^{-1}(f(x)\delta_{t=0}) = \mathbf{1}_{\{t \geq 0\}} e^{t\Delta} f(x)$$

We define  $H^{-\frac{1}{2}}$  similarly:

$$H^{-\frac{1}{2}}f := \frac{1}{(2\pi)^3} \int \frac{1}{(i\tau + |\xi|^2)^{\frac{1}{2}}} e^{i(t\tau + x \cdot \xi)} \tilde{f}(\tau, \xi) d\tau d\xi \quad (3.2)$$

Here we use the principal square root of the complex-valued function  $i\tau + |\xi|^2$  by taking the positive real axis as the branch cut. As the above symbol is still holomorphic for  $\tau$  in the lower half-space, it follows that its kernel is also supported in  $t \geq 0$ . In what follows all these operators are applied only to functions supported on positive time intervals.

Using (2.2), we can rewrite  $A_t$  as

$$A_t = -H^{-1}((Q_{12}(\bar{\phi}, \phi))) - H^{-1}(\partial_1(A_2|\phi|^2)) + H^{-1}(\partial_2(A_1|\phi|^2)), \quad (3.3)$$

where  $Q_{12}(\phi, \bar{\phi}) := \text{Im}(\partial_1\phi\partial_2\bar{\phi} - \partial_2\phi\partial_1\bar{\phi})$ .

Similarly, by (2.3) and initial condition (2.5), we can rewrite  $A_x$  as follows:

$$\begin{aligned} A_1 &= H^{-1}A_1(0) - H^{-1}[\text{Re}(\bar{\phi}\partial_2\phi) + \text{Im}(\bar{\phi}\partial_2\phi)] - H^{-1}(A_2|\phi|^2) \\ A_2 &= H^{-1}A_2(0) + H^{-1}[\text{Re}(\bar{\phi}\partial_1\phi) + \text{Im}(\bar{\phi}\partial_1\phi)] + H^{-1}(A_1|\phi|^2) \end{aligned} \quad (3.4)$$

Here

$$A_1(0) = \frac{1}{2}\Delta^{-1}\partial_2|\phi_0|^2 \quad \text{and} \quad A_2(0) = -\frac{1}{2}\Delta^{-1}\partial_1|\phi_0|^2$$

Our strategy will be to use the contraction principle in the equations (3.4) in order to bound  $A_1$  and  $A_2$  in terms of  $\phi$ , and then to use (3.3) to estimate  $A_t$ . The contraction principle is not applied directly to  $A_1$  and  $A_2$ , but instead to

$$B_1 = H^{-1}(A_2|\phi|^2), \quad B_2 = H^{-1}(A_1|\phi|^2),$$

These functions solve the system

$$\begin{aligned} B_1 &= H^{-1}((H^{-1}A_2(0) + H^{-1}[\text{Re}(\bar{\phi}\partial_1\phi) + \text{Im}(\bar{\phi}\partial_1\phi)])|\phi|^2) + H^{-1}(B_2|\phi|^2) \\ B_2 &= H^{-1}((H^{-1}A_1(0) - H^{-1}[\text{Re}(\bar{\phi}\partial_2\phi) + \text{Im}(\bar{\phi}\partial_2\phi)])|\phi|^2) - H^{-1}(B_1|\phi|^2) \end{aligned} \quad (3.5)$$

We observe here that the first components of  $A_1$  and  $A_2$  depend only on the initial data; therefore they effectively act almost as stationary electromagnetic potentials for the linear Schrödinger equation. The difficulty is that even if  $\phi_0$  is localized, both of these components have only  $|x|^{-1}$  decay at infinity, which in general would make them nonperturbative long range potentials. Fortunately  $A_1(0)$  and  $A_2(0)$  are not independent, and taking into account their interrelation will allow us to still treat their effects in the Schrödinger equation as perturbative. However, this ends up causing considerable aggravation in the construction of our function spaces.

Now we turn our attention to the first equation in (1.4), which we expand using (1.1) as

$$(i\partial_t + \Delta)\phi = N(\phi, A) := -2iA_\ell\partial_\ell\phi - i\partial_\ell A_\ell\phi + A_t\phi + A_x^2\phi - g|\phi|^2\phi \quad (3.6)$$

where  $A_t$ ,  $A_1$  and  $A_2$  are given by (3.3) and (3.4). Our plan is to solve this equation perturbatively. However, in order for this to work, we need to use (3.3) and (3.4) to expand the  $A$ 's in the nonlinearity and replace them by  $B$ 's. Even this expansion is not sufficient in the case of the first term in  $N(\phi, A)$ , for which we need to expand once more. Eventually this leads to an expression of the form

$$N(\phi, A) = L\phi + N_{3,1} + N_{3,2} + N_{3,3} + N_{5,1} + N_{5,2} + \sum_{j=0}^7 E_j$$

where the terms above are as follows:

1.  $L$  contains the linear terms in  $\phi$ , which arise from the first term in  $N$  and the first term in the expansion of  $A$ . It has the form

$$L\phi = iQ_{12}(C, \phi), \quad C = H^{-1}\Delta^{-1}|\phi_0|^2 \quad (3.7)$$

where

$$Q_{12}(C, \phi) = \partial_1 C \partial_2 \phi - \partial_2 C \partial_1 \phi$$

As mentioned before, this term significantly affects our function spaces constructions.

2. The terms  $N_{3,1}$ ,  $N_{3,2}$  and  $N_{3,3}$  are the cubic terms in  $\phi$ , described as follows:

$$N_{3,1} = H^{-1}(\bar{\phi} \partial_1 \phi) \partial_2 \phi - H^{-1}(\bar{\phi} \partial_2 \phi) \partial_1 \phi, \quad (3.8)$$

originates from the first term in  $N$  and the second term in  $A$ . The “null” structure in  $N_{3,1}$  is crucial in our estimates.

$$N_{3,2} = H^{-1}(Q_{12}(\bar{\phi}, \phi)) \phi, \quad (3.9)$$

originates from the second term in  $N$  and the second term in  $A$ , and also from the third term in  $A$  and the first term in  $A_t$ . This also exhibits a null structure that we take advantage of.

$$N_{3,3} = |\phi|^2 \phi \quad (3.10)$$

is the contribution of the last term in  $N$ .

3. The terms  $N_{5,1}$ ,  $N_{5,2}$  and  $N_{5,3}$  are the quintic terms in  $\phi$ , described as follows:

$$N_{5,1} = H^{-1}(H^{-1}(\bar{\phi} \partial \phi) |\phi|^2) \partial \phi \quad (3.11)$$

occurs in the reexpansion of  $A_1$  and  $A_2$  in the first term in  $N$ .

$$N_{5,2} = H^{-1}(\bar{\phi} \partial \phi) H^{-1}(\bar{\phi} \partial \phi) \phi \quad (3.12)$$

occurs in the fourth term in  $N$ , corresponding to the second term in  $A_1, A_2$ .

$$N_{5,3} = H^{-1} \partial (H^{-1}(\bar{\phi} \partial \phi) |\phi|^2) \phi \quad (3.13)$$

occurs in the third term in  $N$ , corresponding to the second term in  $A_1, A_2$  arising in the  $A_t$  expression.

In all these terms it is neither important which spatial derivatives are applied nor where the bar goes in  $\bar{\phi} \partial \phi$ . While they look somewhat different, in the proofs of the multilinear estimates they turn out to be essentially equivalent by a duality argument.

4. The “error” terms are those which can be estimated in a relatively simpler manner. We begin with the multilinear terms containing the data for  $A_x$ , namely

$$E_1 = H^{-1}(H^{-1} A_x(0) |\phi|^2) \partial \phi \quad (3.14)$$

from the reexpansion of the first term in  $N$ ,

$$E_2 = H^{-1} \partial (H^{-1} A_x(0) |\phi|^2) \phi \quad (3.15)$$

from the second and third terms in  $N$ ,

$$E_3 = H^{-1} A_x(0) H^{-1}(\bar{\phi} \partial \phi) \phi \quad (3.16)$$

from the fourth term in  $N$ ,

$$E_4 = (H^{-1} A_x(0))^2 \phi + H^{-1} A_x(0) B \phi + B^2 \phi. \quad (3.17)$$

Finally we conclude with the remaining terms involving  $B$ ,

$$E_5 = H^{-1}(B |\phi|^2) \partial \phi \quad (3.18)$$

from the reexpansion of the first term in  $N$ ,

$$E_6 = H^{-1} \partial (B |\phi|^2) \phi \quad (3.19)$$

from the second and third term in  $N$ ,

$$E_7 = H^{-1}(\bar{\phi} \partial \phi) B \phi \quad (3.20)$$

from the fourth term in  $N$ .

#### 4. FUNCTION SPACES

In this section we define function spaces as in [16, 9], but with some suitable adaptations to the problem at hand. Spaces similar to those in [16, 9] have been used to obtain critical results in different problems [8, 10]. We refer the reader to [8, §2] for detailed proofs of the basic properties of  $U^p, V^p$  spaces.

For a unit vector  $\mathbf{e} \in \mathbb{S}^1$ , we denote by  $H_{\mathbf{e}}$  its orthogonal complement in  $\mathbb{R}^2$  with the induced measure. Define the lateral spaces  $L_{\mathbf{e}}^{p,q}$  with norms

$$\|f\|_{L_{\mathbf{e}}^{p,q}} = \left( \int_{\mathbb{R}} \left( \int_{H_{\mathbf{e}} \times \mathbb{R}} |f(x\mathbf{e} + x', t)|^q dx' dt \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}}$$

with the usual modifications when  $p = \infty$  or  $q = \infty$ .

Define the operator  $P_{N,\mathbf{e}}$  by the Fourier multiplier  $\xi \rightarrow \psi_N(\xi \cdot \mathbf{e})$ , where  $\psi_N$  has symbol  $\psi_N(\xi)$  given by (4.8). The following smoothing estimate plays an important role in our analysis.

**Lemma 4.1** (Local smoothing [11, 12]). *Let  $f \in L^2(\mathbb{R}^2)$ ,  $N \in 2^{\mathbb{Z}}$ ,  $N \geq 1$ , and  $\mathbf{e} \in \mathbb{S}^1$ . Then*

$$\|e^{it\Delta} P_{N,\mathbf{e}} f\|_{L_{\mathbf{e}}^{\infty,2}} \lesssim N^{-\frac{1}{2}} \|f\|_{L^2} \quad (4.1)$$

Also recall the well-known Strichartz estimates.

**Lemma 4.2** (Strichartz estimates [18, 20]). *Let  $(q, r)$  be any admissible pair of exponents, i.e.  $\frac{1}{q} + \frac{1}{r} = \frac{1}{2}$  and  $(q, r) \neq (2, \infty)$ . Then we have the homogeneous Strichartz estimate*

$$\|e^{it\Delta} f\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^2)} \lesssim \|f\|_{L_x^2(\mathbb{R}^2)} \quad (4.2)$$

**4.1.  $U^p$  and  $V^p$  spaces.** Throughout this section let  $H$  be a separable Hilbert space over  $\mathbb{C}$ . Let  $\mathcal{Z}$  be the set of finite partitions  $-\infty \leq t_0 < t_1 < \dots < t_K \leq \infty$  of the real line. If  $t_K = \infty$  and  $v : \mathbb{R} \rightarrow H$ , then we adopt the convention that  $v(t_K) := 0$ . Let  $\chi_I : \mathbb{R} \rightarrow \mathbb{R}$  denote the (sharp) characteristic function of a set  $I \subset \mathbb{R}$ .

**Definition 4.3.** Let  $1 \leq p < \infty$ . For any  $\{t_k\}_{k=0}^K \in \mathcal{Z}$  and  $\{\phi_k\}_{k=0}^{K-1} \subset H$  with  $\sum_{k=0}^{K-1} \|\phi_k\|_H^p = 1$ , we call the function  $a : \mathbb{R} \rightarrow H$  defined by

$$a = \sum_{k=1}^K \chi_{[t_{k-1}, t_k)} \phi_{k-1}$$

a  $U^p$ -atom. We define the atomic space  $U^p(\mathbb{R}, H)$  as the set of all functions  $u : \mathbb{R} \rightarrow H$  admitting a representation

$$u = \sum_{j=1}^{\infty} \lambda_j a_j \text{ for } U^p\text{-atoms } a_j, \{\lambda_j\} \in \ell^1$$

and endow it with the norm

$$\|u\|_{U^p} := \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| : u = \sum_{j=1}^{\infty} \lambda_j a_j, \lambda_j \in \mathbb{C}, a_j \text{ a } U^p\text{-atom} \right\} \quad (4.3)$$

*Remark 4.4.* The spaces  $U^p(\mathbb{R}, H)$  are Banach spaces and we observe that  $U^p(\mathbb{R}, H) \hookrightarrow L^\infty(\mathbb{R}; H)$ . Every  $u \in U^p(\mathbb{R}, H)$  is right-continuous. On occasion the space  $U^p$  is defined in a restricted fashion by requiring that  $t_0 > -\infty$ . Then  $u$  tends to 0 as  $t \rightarrow -\infty$ . The only difference between the two definitions is in whether or not one adds constant functions to  $U^p$ .

**Definition 4.5.** Let  $1 \leq p < \infty$ , We define  $V^p(\mathbb{R}, H)$  as the space of all functions  $v : \mathbb{R} \rightarrow H$  such that

$$\|v\|_{V^p} := \sup_{\{t_k\}_{k=0}^K \in \mathcal{Z}} \left( \sum_{k=1}^K \|v(t_k) - v(t_{k-1})\|_H^p \right)^{\frac{1}{p}} \quad (4.4)$$

is finite.

*Remark 4.6.* We require that two  $V^p$  functions be equal if they are equal in the sense of distributions. Since such functions have at most countably many discontinuous points in time, we adopt the convention that all  $V^p$  functions are right continuous, i.e., we assume we always work with the unique right-continuous representative from the equivalence class.

The spaces  $V^p(\mathbb{R}, H)$  are Banach spaces and satisfy

$$U^p(\mathbb{R}, H) \hookrightarrow V^p(\mathbb{R}, H) \hookrightarrow U^q(\mathbb{R}, H) \hookrightarrow L^\infty(\mathbb{R}; H), \quad p < q \quad (4.5)$$

We denote by  $DU^p$  the space of distributional derivatives of  $U^p$  functions. Then we have the following very useful duality property:

**Lemma 4.7.** *The following duality holds*

$$(DU^p)^* = V^{p'}, \quad 1 \leq p < \infty \quad (4.6)$$

*with respect to a duality relation that extends the standard  $L^2$  duality.*

We refer the reader to [8] for a more detailed discussion.

We also record a useful interpolation property of the spaces  $U^p$  and  $V^p$  (cf. [8, Proposition 2.20]).

**Lemma 4.8.** *Let  $q_1, q_2 > 2$ ,  $E$  be a Banach space and*

$$T : U^{q_1} \times U^{q_2} \rightarrow E$$

*a bounded bilinear operator with  $\|T(u_1, u_2)\|_E \leq C \prod_{j=1}^2 \|u_j\|_{U^{q_j}}$ . In addition, assume that there exists  $C_2 \in (0, C]$  such that the estimate  $\|T(u_1, u_2)\|_E \leq C_2 \prod_{j=1}^2 \|u_j\|_{U^2}$  holds true. Then  $T$  satisfies the estimate*

$$\|T(u_1, u_2)\|_E \lesssim C_2 \left( \ln \frac{C}{C_2} + 1 \right)^2 \prod_{j=1}^2 \|u_j\|_{V^2}, \quad u_j \in V^2, \quad j = 1, 2.$$

*Proof.* The proof is the same as that in [10, Lemma 2.4]. For fixed  $u_2$ , let  $T_1 u := T(u, u_2)$ . Then we have that

$$\|T_1 u\|_E \leq D_1 \|u\|_{U^{q_1}} \quad \text{and} \quad \|T_1 u\|_E \leq D'_1 \|u\|_{U^2}.$$

Here  $D_1 = C \|u_2\|_{U^{q_2}}$ ,  $D'_1 = C_2 \|u_2\|_{U^2}$ .

From the fact that  $\|u_2\|_{U^{q_j}} \leq \|u_2\|_{U^2}$  and [8, Proposition 2.20], we obtain

$$\|T(u_1, u_2)\|_E = \|T_1 u_1\|_E \lesssim C_2 \left( \ln \frac{C}{C_2} + 1 \right) \|u_1\|_{V^2} \|u_2\|_{U^2} \quad (4.7)$$



Then we can repeat the argument by fixing  $u_1$ , using estimate (4.7), and

$$\|T(u_1, u_2)\|_E \leq C \prod_{j=1}^2 \|u_j\|_{U^{q_j}} \leq C \|u_1\|_{V^2} \|u_2\|_{U^{q_j}}$$

□

Let  $\psi : \mathbb{R} \rightarrow [0, 1]$  be a smooth even function compactly supported in  $[-2, 2]$  and equal to 1 on  $[-1, 1]$ . For dyadic integers  $N \geq 1$ , set

$$\psi_N(\xi) = \psi\left(\frac{|\xi|}{N}\right) - \psi\left(\frac{2|\xi|}{N}\right), \quad \text{for } N \geq 2 \quad \text{and} \quad \psi_1(\xi) = \psi(|\xi|). \quad (4.8)$$

For each such  $N \geq 1$ , define the frequency localization operator  $P_N : L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)$  as the Fourier multiplier with symbol  $\psi_N$ . Moreover, let  $P_{\leq N} := \sum_{1 \leq M \leq N} P_M$ . We set  $u_N := P_N u$  for short.

We now introduce  $U^p, V^p$ -type spaces that are adapted to the linear Schrödinger flow.

**Definition 4.9.** For  $s \in \mathbb{R}$ , let  $U_\Delta^p H^s$  (resp.  $V_\Delta^p H^s$ ) be the space of all functions  $u : \mathbb{R} \rightarrow H^s(\mathbb{R}^2)$  such that  $t \mapsto e^{-it\Delta} u(t)$  is in  $U^p(\mathbb{R}, H^s)$  (resp.  $V^p(\mathbb{R}, H^s)$ ), with respective norms

$$\|u\|_{U_\Delta^p H^s} = \|e^{-it\Delta} u\|_{U^p(\mathbb{R}, H^s)}, \quad \|u\|_{V_\Delta^p H^s} = \|e^{-it\Delta} u\|_{V^p(\mathbb{R}, H^s)} \quad (4.9)$$

*Remark 4.10.* The embeddings in Remark 4.6 and Lemma 4.8 naturally extend to the spaces  $U_\Delta^p H^s$  and  $V_\Delta^p H^s$ .

To clarify the roles of the  $U_\Delta^2, V_\Delta^2$  spaces, we introduce the  $X^{0,b}$ -type spaces defined via the norms

$$\|u\|_{\dot{X}^{0, \frac{1}{2}, 1}} = \sum_{\vartheta} \left( \int_{|\tau - \xi^2| = \vartheta} |\tilde{u}(\tau, \xi)|^2 |\tau - \xi^2| d\xi d\tau \right)^{\frac{1}{2}}$$

and

$$\|u\|_{\dot{X}^{s, \frac{1}{2}, \infty}} = \sup_{\vartheta} \left( \int_{|\tau - \xi^2| = \vartheta} |\tilde{u}(\tau, \xi)|^2 |\tau - \xi^2| d\xi d\tau \right)^{\frac{1}{2}}$$

For these spaces we have the embeddings [16]

$$\dot{X}^{0, \frac{1}{2}, 1} \subset U_\Delta^2 L^2 \subset V_\Delta^2 L^2 \subset \dot{X}^{0, \frac{1}{2}, \infty} \quad (4.10)$$

From these inclusions we can conclude that the  $U_\Delta^2$  and  $V_\Delta^2$  norms are equivalent when restricted in modulation to a single dyadic scale.

Another straightforward consequence of the definitions (see for instance [8, Proposition 2.19]) is that one can extend the local smoothing estimate and Strichartz estimates to general  $U_\Delta^p$  functions:

$$\|e^{it\Delta} P_{N, \mathbf{e}} f\|_{L_{\mathbf{e}}^{\infty, 2}} \lesssim N^{-\frac{1}{2}} \|f\|_{U_\Delta^2} \quad (4.11)$$

$$\|e^{it\Delta} f\|_{L_t^q L_x^r(\mathbb{R} \times \mathbb{R}^2)} \lesssim \|f\|_{U_\Delta^p} \quad (4.12)$$

Here  $(q, r)$  is any admissible pair of exponents and  $p := \min(q, r)$ .

Finally, in the case of free solutions for the Schrödinger equation we can easily do orthogonal frequency decompositions. For the  $U_\Delta^2$  and  $V_\Delta^2$  functions we have the following partial substitute:

**Lemma 4.11.** *Let  $1 = \sum_R P_R(D)$  be a locally finite partition of unity in frequency, with uniformly bounded symbols. Then we have the dual bounds*

$$\sum_R \|P_R u\|_{U_\Delta^2}^2 \lesssim \|u\|_{U_\Delta^2}^2$$

respectively

$$\|\sum_R P_R f_R\|_{V_\Delta^2}^2 \lesssim \sum_R \|f_R\|_{V_\Delta^2}^2$$

The proof is straightforward and is left for the reader.

**4.2. Lateral  $U^p$  and  $V^p$  spaces.** Unfortunately the above function spaces are insufficient for closing the multilinear estimates in our problem. Instead we also need to define the lateral  $U^p$  and  $V^p$  spaces.

Given a unit vector  $\mathbf{e} \in \mathbb{S}^1$ , we consider orthonormal coordinates  $(\xi_{\mathbf{e}}, \xi'_{\mathbf{e}})$  with  $\xi_{\mathbf{e}} = \xi \cdot \mathbf{e}$ . Then we define the Fourier region

$$A_{\mathbf{e}} = \{(\tau, \xi) \in \mathbb{R}^2; \xi \cdot \mathbf{e} > \frac{1}{4}|\xi|, |\tau - \xi^2| < \frac{1}{32}\xi^2\}$$

In this region we have

$$\tau - \xi_{\mathbf{e}}'^2 \geq \frac{1}{64}(|\tau| + \xi_{\mathbf{e}}'^2) \quad (4.13)$$

and therefore can factor the symbol of the Schrödinger operator:

$$\xi^2 - \tau = (\xi_{\mathbf{e}} + \sqrt{\tau - \xi_{\mathbf{e}}'^2})(\xi_{\mathbf{e}} - \sqrt{\tau - \xi_{\mathbf{e}}'^2}) \approx |\xi|(\xi_{\mathbf{e}} - \sqrt{\tau - \xi_{\mathbf{e}}'^2})$$

Hence instead of considering the forward Schrödinger evolution we can work with the lateral flow

$$\partial_{\mathbf{e}} - iL_{\mathbf{e}}, \quad L_{\mathbf{e}} = \sqrt{-i\partial_t - \partial_{\mathbf{e}}'^2}$$

for functions frequency localized in the region (4.13). We denote the corresponding  $U^p$  and  $V^p$  function spaces by  $U_{\mathbf{e}}^p$  and  $V_{\mathbf{e}}^p$ , respectively.

Now we are ready to define the nonlinear component of our function spaces, namely  $U^{2,\sharp}$  and  $V^{2,\sharp}$ . For that we need some multipliers, denoted by  $P_{\mathbf{e}}$ , adapted to the regions  $A_{\mathbf{e}}$ . The space  $U^{2,\sharp}$  is given by

$$U^{2,\sharp} = U_\Delta^2 + |D|^{-\frac{1}{2}} \Sigma_{\mathbf{e}} P_{\mathbf{e}} U_{\mathbf{e}}^2 \quad (4.14)$$

In other words it can be thought of as an atomic space where the atoms are normalized  $U_\Delta^2$  functions and normalized  $U_{\mathbf{e}}^2$  functions.

The space  $V^{2,\sharp}$  is given by the norm

$$\|\phi\|_{V^{2,\sharp}} = \|\phi\|_{V_\Delta^2} + \sup_{\mathbf{e}} \| |D|^{\frac{1}{2}} P_{\mathbf{e}} \phi \|_{V_{\mathbf{e}}^2} \quad (4.15)$$

By  $U_k^{2,\sharp}$ , respectively  $V_k^{2,\sharp}$ , we denote the corresponding spaces of functions which are localized at frequency  $2^k$ .

The main properties of these spaces are summarized in the following

**Proposition 4.12.** *The spaces  $U^{2,\sharp}$  and  $V^{2,\sharp}$  defined above have the following properties:*

a) *Inclusion:*

$$U^{2,\sharp} \subset V^{2,\sharp} \quad (4.16)$$

b) *Duality*:

$$[(i\partial_t - \Delta)U^{2,\sharp}]^* = V^{2,\sharp} \quad (4.17)$$

c) *Truncation*. For any time interval  $I$  we have

$$\chi_I : U^{2,\sharp} \rightarrow V^{2,\sharp} \quad (4.18)$$

We postpone the proof of this result for later in the section. Part (a) is a special case of (c) when  $I = \mathbb{R}$ . Part (b) is a direct consequence of the duality result in Lemma 4.7. Part (c) is proved in Lemma 4.15.

**4.3. The  $l_k^2$  spatial structure.** We need one additional structural layer to overlay on top of the  $U^2$  and  $V^2$  structure, which has to do with the fact that we are seeking to solve the problem locally in time. Thus all the estimates we will have to prove apply to functions which are localized in time to a compact interval. Within such an interval, waves at frequency  $2^k$  travel a distance of  $O(2^k)$ , with rapidly decreasing tails farther out. Thus if we partition the space into  $2^k$  sized squares, the interaction of separated squares is negligible. This leads us to introduce a local in time partition of unity

$$1 = \sum_{m \in \mathbb{Z}^2} \chi_k^m(x, t), \quad \chi_k^m(x, t) = \chi_0(t) \chi(2^{-k}x - m)$$

and corresponding norms

$$\|\phi\|_{l_k^2 U^{2,\sharp}}^2 = \sum_{m \in \mathbb{Z}^2} \|\chi_k^m(x, t) \phi\|_{U^{2,\sharp}}^2$$

We similarly define the  $l_k^2 V^{2,\sharp}$  and  $l_k^2 DU^{2,\sharp}$  norms. To relate these norms with the previous ones we have the following:

**Proposition 4.13.** a) For all  $\phi \in U_k^{2,\sharp}$  localized at frequency  $2^k$  we have

$$\|\phi\|_{l_k^2 U^{2,\sharp}} \lesssim \|\phi\|_{U^{2,\sharp}} \quad (4.19)$$

b) For all  $\phi$  localized at frequency  $2^k$  and with compact support in time we have

$$\|\phi\|_{V^{2,\sharp}} \lesssim \|\phi\|_{l_k^2 V^{2,\sharp}} \quad (4.20)$$

**4.4. The angular spaces.** The above spaces suffice in order to treat the nonlinear part of  $N(\phi, A)$ . However, for the linear part  $L$  we need an entirely different type of structure. To set the notation, we denote the angular derivative centered at  $x_0$  by

$$\nabla_{x_0} = (x - x_0) \wedge \partial_x$$

We also set

$$\langle x - x_0 \rangle_k = (2^{-2k} + (x - x_0)^2)^{\frac{1}{2}}$$

Let  $\sigma > 0$  be a fixed constant. For  $x_0 \in \mathbb{R}^2$  and  $k \in \mathbb{Z}$  we define the space  $X_k^{x_0, \sigma}$  with norm

$$\|\phi\|_{X_k^{x_0, \sigma}} = 2^{k(\frac{1}{2} - \sigma)} \|\langle x - x_0 \rangle_k^{-\frac{1}{2} - \sigma} \langle \nabla \rangle^\sigma \phi\|_{L^2} \quad (4.21)$$

as well as the smaller space  $X_k^{x_0, \sigma, \sharp}$  with norm

$$\|\phi\|_{X_k^{x_0, \sigma, \sharp}} = \|\phi(0)\|_{L^2} + \|(i\partial_t - \Delta)\phi\|_{X_k^{x_0, \sigma, *}} \quad (4.22)$$

We further define

$$X_k^\sigma = \bigcap_{x_0 \in \mathbb{R}^2} X_k^{x_0, \sigma}, \quad X_k^{\sigma, *} = \sum_{x_0 \in \mathbb{R}^2} X_k^{x_0, \sigma, *}, \quad X_k^{\sigma, \sharp} = \sum_{x_0 \in \mathbb{R}^2} X_k^{x_0, \sigma, \sharp}, \quad (4.23)$$

where the first space is the dual of the second.

These spaces are used for frequency  $2^k$  solutions to the Schrödinger equation. Their main properties are stated in the following

**Proposition 4.14.** *The spaces defined above, restricted to frequency  $2^k$  functions, have the following properties:*

a) *Solvability:*

$$\|\phi\|_{X_k^{\sigma,\sharp}} \lesssim \|\phi(0)\|_{L^2} + \|(i\partial_t - \Delta)\phi\|_{X_k^{\sigma,*}} \quad (4.24)$$

b) *Moving centers:*

$$\|\phi\|_{X_k^\sigma} + \|\phi\|_{L^\infty L^2} \lesssim \|\phi\|_{X_k^{\sigma,\sharp}} \quad (4.25)$$

c) *Nesting:*

$$\|\phi\|_{X_k^{\sigma_1}} \lesssim \|\phi\|_{X_k^{\sigma_2}}, \quad \sigma_2 < \sigma_1 \quad (4.26)$$

In this result  $k$  is a scaling parameter and can be set to 0. Part (a) is a straightforward consequence of the definitions. However, part (b) is far less trivial, and requires two separate estimates. First, for fixed  $x_0$  we need to show that

$$\|\phi\|_{X_k^{x_0,\sigma}} + \|\phi\|_{L^\infty L^2} \lesssim \|\phi\|_{X_k^{x_0,\sigma,\sharp}} \quad (4.27)$$

which is done in Lemma 4.16.

Secondly, for  $x_1 \neq x_0$  we need to show that

$$\|\phi\|_{X_k^{x_1,\sigma}} \lesssim \|\phi\|_{X_k^{x_0,\sigma}} + \|\phi\|_{X_k^{x_0,\sigma,\sharp}} \quad (4.28)$$

This is achieved in Lemma 4.18. Part (c) follows from the similar property for fixed  $x_0$ , which is straightforward in view of the frequency localization.

**4.5. Dyadic norms and the main function spaces.** Finally, we are ready to set up the global function spaces where we solve the Chern-Simons-Schrödinger problem. For the solutions at frequency  $2^k$  we use two spaces. The stronger norm  $X_k^\sharp$  represents the space where the solutions actually lie and is given by

$$X_k^\sharp = l_k^2 U_k^{2,\sharp} + X_k^{\sigma,\sharp} \quad (4.29)$$

Here  $0 < \sigma < \frac{1}{2}$  is a fixed constant.

However, this is a sum type space and so multilinear estimates would be quite cumbersome, with many cases. Furthermore, the above space is not stable with respect to time truncations. Instead we also introduce a weaker topology

$$X_k = l_k^2 V_k^{2,\sharp} \cap X_k^\sigma \quad (4.30)$$

For the inhomogeneous term in the equation we have the space  $Y_k$  which has  $X_k$  as its dual,

$$Y_k = l_k^2 D U_k^{2,\sharp} + X_k^{\sigma,*} \quad (4.31)$$

The main result concerning our function spaces is in the following

**Theorem 1.** *The following properties are valid for frequency  $2^k$  functions:*

a) *Linear estimate:*

$$\|\phi\|_{X_k^\sharp} \lesssim \|\phi(0)\|_{L^2} + \|(i\partial_t - \Delta)\phi\|_{Y_k} \quad (4.32)$$

b) *Weaker norm:*

$$\|\phi\|_{X_k} \lesssim \|\phi\|_{X_k^\sharp} \quad (4.33)$$

c) *Time truncation:*

$$\|\chi_I \phi\|_{X_k} \lesssim \|\phi\|_{X_k^\sharp}, \quad I \subset \mathbb{R} \quad (4.34)$$

d) *Duality:*

$$Y_k^* = X_k \quad (4.35)$$

Part (a) is a direct consequence of the preceding three propositions. The estimate in (b) is a special case of (c). Part (c) also follows in part from the two preceding propositions. However, we still need to address the cross embeddings,

$$\chi_I l_k^2 U_k^{2,\sharp} \subset X_k^\sigma$$

respectively

$$\chi_I X_k^{\sigma,\sharp} \subset l_k^2 V_k^{2,\sharp}$$

The truncation in the first embedding can be harmlessly dropped as it is bounded on  $X_k^\sigma$ . It remains to show the embedding

$$l_k^2 U_k^{2,\sharp} \subset X_k^\sigma \quad (4.36)$$

which is proved in Lemma 4.17. The truncation in the second embedding can also be dropped. To see this recall that  $X_k^{\sigma,\sharp} \subset L^\infty L^2$ . This allows us to freely replace arbitrary functions  $u \in X_k^{\sigma,\sharp}$  by solutions to the homogeneous equation outside  $I$ . But then  $\chi_I u - u \in U_\Delta^2$  and we can use (4.16). Once  $\chi_I$  is dropped, using again  $X_k^{\sigma,\sharp} \subset L^\infty L^2$ , the problem reduces to

$$X_k^{\sigma,*} \subset l_k^2 D V_k^{2,\sharp}$$

which follows from (4.36) by duality.

In this article we work with  $H^s$  initial data. Correspondingly, we define the spaces  $X^{s,\sharp}$  and  $X^s$  for solutions, respectively  $Y^s$  for the nonlinearity, by

$$\|\phi\|_{X^{s,\sharp}}^2 = \sum_{k \geq 0} 2^{2sk} \|P_k \phi\|_{X_k^\sharp}^2 \quad (4.37)$$

$$\|\phi\|_{X^s}^2 = \sum_{k \geq 0} 2^{2sk} \|P_k \phi\|_{X_k}^2 \quad (4.38)$$

$$\|f\|_{Y^s}^2 = \sum_{k \geq 0} 2^{2sk} \|P_k f\|_{Y_k}^2 \quad (4.39)$$

where  $P_0$  includes all frequencies less than 1.

**4.6. The nonlinearity  $N(\phi, A)$ .** Here we turn our attention to the nonlinear equation

$$(i\partial_t - \Delta)\phi = N(\phi, A), \quad \phi(0) = \phi_0, \quad A = A(\phi),$$

where  $A(\phi)$  is obtained by solving (3.3)-(3.4). We seek to solve this equation for positive  $t$  and locally in time; therefore we can harmlessly insert a cutoff function  $\chi = \chi_{[0,1]}$  in time and solve instead the modified equation

$$(i\partial_t - \Delta)\phi = N(\chi\phi, A), \quad \phi(0) = \phi_0, \quad A = A(\chi\phi) \quad (4.40)$$

Any global solution to this modified equation will solve the original equation in the time interval  $[0, 1]$ .

We use the  $H^s$  version of the linear estimate (4.32) to solve this equation in the space  $X^{s,\sharp}$  using the contraction principle. Thus we need to show that we have a small Lipschitz constant for the map

$$X^{s,\sharp} \ni \phi \rightarrow N(\chi\phi, A) \in Y^s, \quad A = A(\chi\phi)$$

We subdivide this problem into two completely different problems, which correspond to the decomposition of  $N(\chi\phi, A)$  into a linear and a nonlinear part. To estimate the linear part  $L(\chi\phi)$  we will use the  $H^s$  version of the embedding (4.34) and select only the  $X^{\sigma,s}$  part of the  $X^s$  norm, neglecting the  $l^2$  structure. Then we can drop the cutoff  $\chi$ , and it remains to prove the bound

$$\|L\phi\|_{X^{\sigma,s,*}} \lesssim \|\phi\|_{X^{\sigma,s}} \|\phi(0)\|_{H^s}^2 \quad (4.41)$$

This is achieved in Section 5, Proposition 5.1.

To estimate the nonlinear part  $Nl(\chi\phi, A) = N(\chi\phi, A) - L\chi\psi$  we seek to prove

$$X^{s,\sharp} \ni \phi \rightarrow Nl(\chi\phi, A) \in Y^s, \quad A = A(\chi\phi)$$

Retaining only the  $U^2$ - $V^2$  part of our function spaces, it suffices consider the map

$$l^2V^{2,\sharp,s} \ni \phi \rightarrow Nl(\phi, A) \in l^2DU^{2,\sharp,s}, \quad A = A(\phi)$$

for  $\phi$  localized in time. By duality, this translates to Lipschitz continuity of the form

$$l^2V^{2,\sharp,s} \times l^2V^{2,\sharp,-s} \ni (\phi, \psi) \rightarrow \int Nl(\phi, A) \bar{\psi} \, dxdt \quad (4.42)$$

To prove this we succesively consider all the terms in  $Nl(\phi, A)$  in Sections 8-10.

**4.7. Linear estimates.** We now proceed to state and prove a collection of linear lemmas which, together, imply the results stated before in this section.

Given an angle  $A$  in  $\mathbb{R}^2$  with opening less than  $\pi$ , we say that a direction  $\mathbf{e}$  is admissible with respect to  $A$  if  $\pm\mathbf{e}^\perp \notin A$ .

For  $k \geq 0$  we define the following subset of Fourier space

$$A_k = \{(\xi, \tau) \in A \times \mathbb{R}; |\xi| \sim 2^k, |\tau - \xi^2| \ll 2^{2k}\}$$

We denote by  $P_{A,k}$  a smooth space-time multiplier with support in  $A_k$ . Then we have

**Lemma 4.15.** *Let  $A$  be an angle in  $\mathbb{R}^2$ ,  $k > 0$ ,  $I$  a time interval and  $\mathbf{e}_1, \mathbf{e}_2$  admissible directions with respect to  $A$ . Then for functions  $f$  that are frequency localized in  $A_k$ , we have*

$$\|P_{A,k}\chi_I f\|_{V_{\mathbf{e}_2}^2} \lesssim \|f\|_{U_{\mathbf{e}_1}^2} \quad (4.43)$$

*with an implicit constant that is uniform with respect to pairs  $\mathbf{e}_1, \mathbf{e}_2$  for which the distances  $\text{dist}(\pm\mathbf{e}_{1,2}, A)$  lie in a compact set away from zero.*

This lemma serves to prove the properties (4.16) and (4.18) in Proposition 4.12. We note that two nontrivial properties are coupled in the statement, namely the embedding  $U_{\mathbf{e}_1}^2 \subset V_{\mathbf{e}_2}^2$  and the time truncation. We further note that the same estimate holds true if either of the two lateral spaces is replaced by the corresponding vertical space. In that case the time truncation can be absorbed into the vertical space and one is left with just the embedding, for which the proof below still applies.

*Proof.* By scaling we can assume that  $k = 0$ . We consider the multiplier  $P$  which selects a small neighborhood of  $A_0$ . Then  $P$  is bounded on both  $V_{\mathbf{e}_2}^2$  and  $U_{\mathbf{e}_1}^2$ , and so it suffices to show that

$$\|P\chi_I P f\|_{V_{\mathbf{e}_2}^2} \lesssim \|f\|_{U_{\mathbf{e}_1}^2} \quad (4.44)$$

Set  $I = [t_0, t_1]$ . We can harmlessly replace  $\chi_I$  by its mollified version  $Q_{\ll 0}\chi_I$  as its high modulation part provides no output.

We first observe that the simpler bound

$$\|Pf\|_{L^{\infty,2}_{\mathbf{e}_2}} + \|Pf\|_{L^\infty L^2} \lesssim \|f\|_{U^2_{\mathbf{e}_1}} \quad (4.45)$$

follows easily by reducing to a  $U^2_{\mathbf{e}_1}$  atom where the free waves associated to each step are supported in a small neighborhood of the intersection of  $A_0$  with the paraboloid. Then we apply either the energy estimate or the lateral energy estimate in the  $\mathbf{e}_2$  direction for each step of that atom.

Then the bound (4.44) reduces to

$$\|(i\partial_t - \Delta)P\chi_I Pf\|_{DV^2_{\mathbf{e}_2}} \lesssim \|f\|_{U^2_{\mathbf{e}_1}}$$

Using the duality between  $DV^2$  and  $U^2$ , this is equivalent to the symmetric bound

$$|Q_R(f, g)| \lesssim \|f\|_{U^2_{\mathbf{e}_1}} \|g\|_{U^2_{\mathbf{e}_2}} \quad (4.46)$$

where

$$Q_R(f, g) = \langle (i\partial_t - \Delta)\chi_I Pf, Pg \rangle = \langle Pf, \chi_I(i\partial_t - \Delta)Pg \rangle$$

This is not entirely symmetric, and so we also introduce its twin

$$Q_L(f, g) = \langle \chi_I(i\partial_t - \Delta)Pf, Pg \rangle$$

Their difference is easy to control. Indeed, we have

$$Q_R(f, g) - Q_L(f, g) = \langle [(i\partial_t - \Delta), \chi_I]Pf, Pg \rangle = \langle i\partial_t \chi_I Pf, Pg \rangle$$

The time derivative of  $\chi_I$  is a sum of two unit bump functions on a unit time interval around  $t_0$ , respectively  $t_1$ . Hence using the energy part of (4.45) we obtain

$$|Q_R(f, g) - Q_L(f, g)| \lesssim \|f\|_{U^2_{\mathbf{e}_1}} \|g\|_{U^2_{\mathbf{e}_2}} \quad (4.47)$$

Given the support of  $P$ , we can rewrite  $Q_L$  and  $Q_R$  in terms of the sideways evolutions for  $f$  and  $g$ :

$$\begin{aligned} Q_L(f, g) &= \langle \chi_I P(D_{\mathbf{e}_1} - L_{\mathbf{e}_1})f, Pg \rangle, \\ Q_R(f, g) &= \langle Pf, \chi_I P(D_{\mathbf{e}_1} - L_{\mathbf{e}_1})g \rangle \end{aligned} \quad (4.48)$$

Here the elliptic factor in the factorization of  $i\partial_t - \Delta$  is included in  $P$ . Thus by a slight abuse of notation we use the same  $P$  for different multipliers with similar size and support.

It suffices to prove (4.46) for atoms. Thus consider  $f$  and  $g$  of the form

$$f = \sum \chi_{[a_i, b_i]}(x \cdot \mathbf{e}_1) f_i, \quad g = \sum \eta_{[c_i, d_i]}(x \cdot \mathbf{e}_2) g_i$$

where  $f_i$  and  $g_i$  are homogeneous waves, frequency localized in a small neighborhood of  $A_1$ , and with

$$\sum_i \|f_i(0)\|_{L^2}^2 \approx 1, \quad \sum_i \|g_i(0)\|_{L^2}^2 \approx 1$$

As  $f_i$  and  $g_i$  are free waves frequency localized near the  $A$  section on the parabola at frequency one, we can measure their energy in an equivalent way at time  $t = 0$ .

Instead of the data at time  $t = 0$ , it is better to describe  $f_i$  in terms of its values at  $x \cdot \mathbf{e}_1 = a_i$  and at  $x \cdot \mathbf{e}_1 = b_i$ . By a slight abuse of notation we denote these two functions by  $f_i(a_i)$  and  $f_i(b_i)$ . We remark that  $f_i(a_i)$  and  $f_i(b_i)$  are related via the sideways evolution and in particular we have

$$\|f_i(a_i)\|_{L^2} = \|f_i(b_i)\|_{L^2}$$

However, it will be convenient to work with both of them together rather than separately.

We observe that it suffices to consider the case when  $b_i - a_i \gg 1$  and  $c_i - d_i \gg 1$ . Indeed, if for instance  $b_i - a_i \lesssim 1$  for all  $i$  then

$$\|f\|_{L^2} \lesssim 1$$

This is easily combined with the following easy consequence of (4.10),

$$\|(i\partial_t - \Delta)Pg\|_{L^2} \lesssim 1,$$

to conclude the argument.

We can also assume without any restriction in generality that  $a_{i+1} - b_i \gg 1$  and  $c_{i+1} - d_i \gg 1$ . To the intervals  $[a_i, b_i]$  we associate bump functions  $\chi_i$  which equal 1 inside the interval and decay rapidly on the unit scale. By  $\eta_i$  we denote similar bump functions associated to  $[c_i, d_i]$ . Set

$$\begin{aligned} B_L^{ij} &:= Q_L(\chi_{[a_i, b_i]}(x \cdot \mathbf{e}_1) f_i, \eta_{[c_j, d_j]}(x \cdot \mathbf{e}_2) g_j) \\ B_R^{ij} &:= Q_R(\chi_{[a_i, b_i]}(x \cdot \mathbf{e}_1) f_i, \eta_{[c_j, d_j]}(x \cdot \mathbf{e}_2) g_j) \end{aligned}$$

We want to be able to use  $Q_L$  and  $Q_R$  interchangeably. For that we estimate the difference

$$B_L^{ij} - B_R^{ij} = \langle i\partial_t \chi_I P \chi_{[a_i, b_i]}(x \cdot \mathbf{e}_1) f_i, P \eta_{[c_j, d_j]}(x \cdot \mathbf{e}_2) g_j \rangle$$

Using the time localization given by  $\partial_t \chi_I$  and the finite speed of propagation in time for waves supported in  $A_0$ , we obtain a localized analogue of (4.47), namely

$$|B_L^{ij} - B_R^{ij}| \lesssim \|\chi_i \eta_j f_i(t_0)\|_{L^2} \|\chi_i \eta_j g_j(t_0)\|_{L^2} + \|\chi_i \eta_j f_i(t_1)\|_{L^2} \|\chi_i \eta_j g_j(t_1)\|_{L^2}$$

By Cauchy-Schwarz this implies that

$$\sum_{i,j} |B_L^{ij} - B_R^{ij}| \lesssim 1 \quad (4.49)$$

which indeed allows us to estimate  $B_L^{ij}$  and  $B_R^{ij}$  interchangeably. Using the representation of  $Q_L$  in (4.48) we have

$$\begin{aligned} B_L^{ij} &:= Q_L(\chi_{[a_i, b_i]}(x \cdot \mathbf{e}_1) f_i, \eta_{[c_j, d_j]}(x \cdot \mathbf{e}_2) g_j) \\ &= \langle f_i(b_i) \delta_{x \cdot \mathbf{e}_1 = b_i} - f_i(a_i) \delta_{x \cdot \mathbf{e}_1 = a_i}, P(\eta_{[c_j, d_j]}(x \cdot \mathbf{e}_2) g_j) \rangle \end{aligned}$$

A symmetric formula holds for  $B_R^{ij}$ . For  $g_j$  we have lateral energy estimates in the  $\mathbf{e}_1$  directions, and  $P$  has a rapidly decreasing kernel. Hence the above expression is bounded by

$$|B_L^{ij}| \lesssim \|\eta_j f_i(b_i)\|_{L^2} \|\eta_j g_j(b_i)\|_{L^2} + \|\eta_j f_i(a_i)\|_{L^2} \|\eta_j g_j(a_i)\|_{L^2} \quad (4.50)$$

In order to complete the proof of (4.46) for atoms we need to distinguish between different interval balances:

**A. Unbalanced intervals:** Either  $b_i - a_i \gg d_j - c_j$  or  $b_i - a_i \ll d_j - c_j$ . In this case we will prove that

$$\begin{aligned} \min\{|B_L^{ij}|, |B_R^{ij}|\} &\lesssim (\|\tilde{\eta}_j(x \cdot \mathbf{e}_2) f_i(b_i)\|_{L^2} + \|\tilde{\eta}_j(x \cdot \mathbf{e}_2) f_i(a_i)\|_{L^2}) \\ &\quad (\|\tilde{\chi}_i(x \cdot \mathbf{e}_1) g_j(d_j)\|_{L^2} + \|\tilde{\chi}_i(x \cdot \mathbf{e}_1) g_j(c_j)\|_{L^2}) \end{aligned} \quad (4.51)$$



for some more relaxed bump functions  $\tilde{\eta}_j$  and  $\tilde{\chi}_i$  which share the properties of  $\chi_i$  and  $\eta_j$ . Assuming (4.51) is true, the estimate for the corresponding part of (4.46) easily follows from Cauchy-Schwarz:

$$\begin{aligned} \sum_{i,j} \min\{|B_L^{ij}|, |B_R^{ij}|\} &\lesssim \sum_{i,j} \|\tilde{\eta}_j(x \cdot \mathbf{e}_2) f_i(b_i)\|_{L^2}^2 + \|\tilde{\eta}_j(x \cdot \mathbf{e}_2) f_i(a_i)\|_{L^2}^2 \\ &\quad + \sum_{i,j} \|\tilde{\chi}_i(x \cdot \mathbf{e}_1) g_j(d_j)\|_{L^2}^2 + \|\tilde{\chi}_i(x \cdot \mathbf{e}_1) g_j(c_j)\|_{L^2}^2 \\ &\lesssim \sum_i \|f_i(b_i)\|_{L^2}^2 + \|f_i(a_i)\|_{L^2}^2 + \sum_j \|g_j(d_j)\|_{L^2}^2 + \|g_j(c_j)\|_{L^2}^2 \\ &\lesssim 1 \end{aligned}$$

By symmetry suppose that  $b_i - a_i \gg d_j - c_j$ . Then (4.51) follows from (4.50) due to the propagation estimate

$$\|\eta_j g_j(b_i)\|_{L^2} + \|\eta_j g_j(a_i)\|_{L^2} \lesssim \|\chi_i g_j(d_j)\|_{L^2} + \|\chi_i g_j(c_j)\|_{L^2}$$

To see this it suffices to consider the Schrödinger propagator from the surfaces  $x \cdot \mathbf{e}_1 = a_i, b_i$  to the surfaces  $x \cdot \mathbf{e}_2 = c_j, d_j$ . corresponding to waves which are localized in  $A_0$ . On the one hand, with respect to suitable elliptic multiplier weights, this is an  $L^2$  isometry. On the other hand, its kernel decays rapidly outside a conic neighborhood of the propagation cone associated to  $A_0$ . Hence all that remains to be seen is that the propagation cone of the interval  $\{x \cdot \mathbf{e}_1 = a_i, x \cdot \mathbf{e}_2 \in [c_j, d_j]\}$  either intersects the line  $x \cdot \mathbf{e}_2 = c_j$  within the interval  $x \cdot \mathbf{e}_1 \in [a_i, b_i]$  or intersects the line  $x \cdot \mathbf{e}_2 = d_j$  within the interval  $x \cdot \mathbf{e}_1 \in [a_i, b_i]$ . But this is a geometric consequence of the unbalanced intervals.

**B. Balanced intervals.** Here we consider the case when  $b_i - a_i \sim d_j - c_j$ . The first observation is that it suffices to consider a fixed dyadic scale  $X$  and assume that

$$b_i - a_i \sim d_j - c_j \sim X$$

The dyadic summation with respect to  $X$  will be straightforward since we have  $l^2$  summability both on the  $f$  and on the  $g$  side.

The simplification that occurs when we fix the interval size is that we are allowed to relax the localization scale in the choice of the functions  $\chi_j$  and  $\eta_j$  in (4.50). Precisely, instead of the rapid decay on the unit scale (dictated by the smallest distance to the next interval) we allow them to decay rapidly on the  $X$  scale, and denote them by  $\chi_i^X$  and  $\eta_j^X$ . This makes the following norms equivalent:

$$\|\eta_j^X f_j(b_i)\|_{L^2} \approx \|\eta_j^X f_j(a_i)\|_{L^2} \approx \|\chi_i^X f_j(c_j)\|_{L^2} \approx \|\chi_i^X f_j(d_j)\|_{L^2}$$

by standard propagation arguments.

By (4.50) this implies the version of (4.51) with the weights  $\chi_i^X$  and  $\eta_j^X$ . The punch line is then in the  $i$  and  $j$  summation argument under (4.51). The bumps  $\chi_i^X$  and  $\eta_j^X$  are wider now, but they are still almost orthogonal since the intervals are now also uniformly spaced at distance  $X$  (or above).

□

Our next lemma serves to prove the estimate (4.27), which is needed for Proposition 4.14. In order to do that we need two more definitions, namely the local energy space (centered at 0)  $LE$  and its dual  $LE^*$ . The  $LE$  space-time norm adapted to frequency-one functions is defined as

$$\|\phi\|_{LE} = \|\phi\|_{L^2(|x| \lesssim 1)} + \sup_{j>0} 2^{-\frac{j}{2}} \|\phi\|_{L^2(|x| \approx 2^j)}$$

Then we have

**Lemma 4.16.** *Let  $s > 0$ . Then for frequency-one functions  $u$  solving  $(i\partial_t - \Delta)u = f_1 + f_2$ , where  $f_1$  has no radial modes, we have the following estimate*

$$\|u\|_{LE} + \|\langle r \rangle^{-\frac{1}{2}-s} \langle \nabla \rangle^s u\|_{L^2} \lesssim \|u(0)\|_{L^2} + \|\langle r \rangle^{\frac{1}{2}+s} \langle \nabla \rangle^{-s} f_1\|_{L^2} + \|f_2\|_{LE^*} \quad (4.52)$$

*Proof.* Our starting point is the standard local energy decay estimate for frequency-one functions, namely

$$\|u\|_{LE} \lesssim \|u(0)\|_{L^2} + \|f\|_{LE^*} \quad (4.53)$$

We expand the function  $u$  in (4.52) in an angular Fourier series. This preserves the frequency localization, and it suffices to prove (4.52) for each such mode separately (with uniform constants).

Our contention is that for a fixed angular mode the bound (4.52) is a direct consequence of (4.53). To see that let  $k \in \mathbb{Z}$  and  $u$  be of the form

$$u_k(t, x) = u_k(r, t) e^{ik\theta}$$

Then we have

$$\langle r \rangle^{-\frac{1}{2}-s} \nabla^s u_k = \langle r \rangle^{-\frac{1}{2}-s} k^s u_k$$

This is easily controlled by the  $LE$  norm of  $u$  for  $r \gtrsim k$ . To deal with smaller  $r$  we need to use the angular localization. Precisely, we claim that

$$\|r^{-\frac{1}{2}-s} P_0 u_k\|_{L^2} \lesssim \|(r+k)^{-\frac{1}{2}-s} u_k\|_{L^2} \quad (4.54)$$

which easily leads to

$$k^s \|\langle r \rangle^{-\frac{1}{2}-s} P_0 u_k\|_{L^2} \lesssim \|u_k\|_{LE}$$

and, by duality,

$$\|P_0 f_k\|_{LE^*} \lesssim k^{-s} \|\langle r \rangle^{\frac{1}{2}+s} f_k\|_{L^2}$$

The last two bounds prove that (4.52) follows from (4.53).

It remains to establish (4.54). The kernel of  $P_0$  is given by a Schwartz function  $\phi$ . Then for  $|x| \lesssim k$  we write

$$P_0 u_k(x) = \int \phi(x-y) u(y) dy = (-k)^{-N} \int \nabla_y^N \phi(x-y) u(y) dy.$$

We have

$$|\nabla_y^N \phi(x-y)| \lesssim \langle x \rangle^N \langle x-y \rangle^{-N}$$

and therefore

$$|P_0 u_k(x)| \lesssim \left( \frac{\langle r \rangle}{|k|} \right)^N \int \langle x-y \rangle^{-N} |u(y)| dy$$

Hence (4.54) easily follows. □

As a consequence of the above lemma we get the following result, which proves the embedding (4.36) needed in Theorem 1.

**Lemma 4.17.** *The following inequality holds for frequency  $2^k$  functions  $u$ :*

$$2^{(\frac{1}{2}-s)k} \|\langle r \rangle_k^{-\frac{1}{2}-s} \langle \nabla \rangle^s u\|_{L^2} \lesssim \|u\|_{U^{2,\sharp}} \quad (4.55)$$

*In addition, if  $u$  is localized in a unit time interval then*

$$2^{(\frac{1}{2}-s)k} \|\langle r \rangle_k^{-\frac{1}{2}-s} \langle \nabla \rangle^s u\|_{L^2} \lesssim \|u\|_{l_k^2 U^{2,\sharp}} \quad (4.56)$$

*Proof.* It suffices to consider  $0 < s < \frac{1}{2}$ . For the vertical  $U_\Delta^2$  space the bound (4.55) is a direct consequence of the previous lemma via the atomic decomposition. It remains to consider a lateral  $U_e^2$  space and a corresponding atom

$$u = \sum \chi_j u_j$$

For each of these atoms we have

$$2^{(\frac{1}{2}-s)k} \|\langle r \rangle_k^{-\frac{1}{2}-s} \langle \nabla \rangle^s u_j\|_{L^2} \lesssim \|u_j(0)\|_{L^2}$$

Thus it would suffice to show that

$$\|\langle r \rangle_k^{-\frac{1}{2}-s} \langle \nabla \rangle^s (\sum \chi_j u_j)\|_{L^2}^2 \lesssim \sum \|\langle r \rangle^{-\frac{1}{2}-s} \langle \nabla \rangle^s u_j\|_{L^2}^2$$

The bound (4.56) also reduces to the same estimate, with the only difference being that the atomic decomposition is now done separately in each  $2^k$  sized spatial cube.

The last bound reduces to an estimate on the unit circle,

$$\|\sum \chi_j u_j\|_{H^s(\mathbb{S}^1)}^2 \lesssim \sum \|u_j\|_{H^s(\mathbb{S}^1)}^2, \quad 0 \leq s < \frac{1}{2}$$

It makes no difference whether this is done on the circle or on the real line. The following argument is for the case of the real line. We begin with a simple observation, namely that

$$\|\chi_j u_j\|_{H^s} \lesssim \|u_j\|_{H^s}$$

Hence we can drop the cutoffs  $\chi_j$  and instead assume that the  $u_j$  have disjoint supports in consecutive intervals  $I_j$ . We use a Littlewood-Paley decomposition, but instead of having sharp Fourier localization it is convenient to choose multipliers  $P_k$  whose kernels have sharp localization in the physical space on the  $2^{-k}$  scale.

To estimate  $P_k(\sum u_j)$ , we split the intervals  $I_j$  into long ( $2^k |I_j| > 1$ ) and short ( $2^k |I_j| < 1$ ). The outputs of long intervals are almost orthogonal,

$$\|P_k(\sum_{I_j \text{ long}} u_j)\|_{L^2}^2 \lesssim \sum_{I_j \text{ long}} \|P_k u_j\|_{L^2}^2$$

It remains to consider the outputs of short intervals. We still have orthogonality at interval separations of  $2^{-j}$ , and so we can write

$$2^{2sk} \|P_k(\sum_{I_j \text{ short}} u_j)\|_{L^2}^2 \lesssim 2^{2sk} \sum_{|I|=2^{-k}} \left( \sum_{I_j \subset 2I} \|P_k u_j\|_{L^2} \right)^2$$

But for short intervals we can use the fact that the kernel of  $P_k$  is a bump function with  $2^{-k}$  sized support and  $2^k$  amplitude to write

$$\|P_k u_j\|_{L^2} \lesssim |I_j|^{\frac{1}{2}} 2^{k/2} \|u_j\|_{L^2} \lesssim |I_j|^{\frac{1}{2}+s} 2^{k/2} \|u_j\|_{H^s}$$

Hence we obtain

$$\begin{aligned} 2^{2sk} \|P_k(\sum_{I_j \text{ short}} u_j)\|_{L^2}^2 &\lesssim \sum_{|I|=2^{-k}} \left( \sum_{I_j \subset 2I} (2^k |I_j|)^{\frac{1}{2}+s} \|u_j\|_{H^s} \right)^2 \\ &\lesssim \sum_{|I|=2^{-k}} \left( \sum_{I_j \subset 2I} 2^k |I_j| \right) \left( \sum_{I_j \subset 2I} (2^k |I_j|)^{2s} \|u_j\|_{H^s}^2 \right) \\ &\lesssim \sum_{2^k |I_j| \leq 1} (2^k |I_j|)^{2s} \|u_j\|_{H^s}^2 \end{aligned}$$

and the  $k$  summation is straightforward. □

Finally, the following lemma allows us to move centers and proves the estimate (4.28), which is needed in Proposition 4.14.:

**Lemma 4.18.** *For  $u$  localized at frequency one solving  $(i\partial_t - \Delta)u = f$  and for any  $x_0 \in \mathbb{R}^2$ , we have*

$$\|\langle x - x_0 \rangle^{-\frac{1}{2}-s} \langle \nabla_{x_0} \rangle^s u\|_{L^2} \lesssim \|u(0)\|_{L^2} + \|\langle r \rangle^{\frac{1}{2}+s} \langle \nabla \rangle^{-s} f\|_{L^2} \quad (4.57)$$

*Proof.* We split  $u$  into several spatial regions depending on the ratio of  $\langle x - x_0 \rangle$  and  $R = \langle x_0 \rangle$ .

(i) The intermediate region,  $A_{med} = \{\langle x - x_0 \rangle \approx R\}$ . In this region it suffices to use the local energy decay

$$\|\langle x - x_0 \rangle^{-\frac{1}{2}-s} \langle \nabla_{x_0} \rangle^s \chi_{med} u\|_{L^2} \lesssim R^{-\frac{1}{2}} \|\chi_{med} u\|_{L^2} \lesssim \|u\|_{LE}$$

and then the previous lemma.

(ii) The inner region,  $A_{in} = \{\langle x - x_0 \rangle \ll R\}$ . Here we compute

$$(i\partial_t - \Delta)(\chi_{in} u) = f_{in} := -2\nabla \chi_{in} \cdot \nabla u - \Delta \chi_{in} \cdot u + \chi_{in} f$$

and use the local energy bound for  $u$  to estimate

$$\|f_{in}\|_{LE^*} \lesssim \|u\|_{LE} + \|\chi_{in} f\|_{LE^*} \lesssim \|u\|_{LE} + \|\langle r \rangle^{\frac{1}{2}+s} \langle \nabla \rangle^{-s} f\|_{L^2},$$

where at the last step we have used the fact that  $\chi_{in}$  is supported in a single dyadic region  $\langle x \rangle \approx R$ . Then we apply Lemma 4.16.

(iii) The outer region,  $A_{out} = \{\langle x - x_0 \rangle \gg R\}$ . Here we use the following estimate which applies for frequency one functions

$$\|\langle x - x_0 \rangle^{-\frac{1}{2}-s} \langle \nabla_{x_0} \rangle^s \chi_{out} u\|_{L^2} \lesssim \|\langle x \rangle^{-\frac{1}{2}-s} \langle \nabla \rangle^s u\|_{L^2} + R^s \|\langle x + R \rangle^{-\frac{1}{2}-s} u\|_{L^2},$$

and estimate the second term on the right by the local energy norm. This in turn is proved by complex interpolation between  $s = 0$  and  $s = 1$  since

$$\nabla - \nabla_{x_0} = x_0 \cdot \nabla$$

□

## 5. THE LINEAR PART OF $N(\phi, A)$

In this section we prove the main estimate (4.41) for the component  $L$  of the nonlinearity, see (3.7). For convenience we restate the full result here:

**Proposition 5.1.** *Let  $C = H^{-1} \Delta^{-1} |\phi_0|^2$ . Then for  $s \geq 0$  we have*

$$\|Q_{12}(C, \phi)\|_{X^{\sigma, s, *}} \lesssim \|\phi\|_{X^{\sigma, s}} \|\phi_0\|_{H^s}^2 \quad (5.1)$$

*Proof.* For  $u_0 = |\phi_0|^2$  we use the multiplicative Sobolev estimate

$$\|u_0\|_{L^1} + \|u_0\|_{B_s^{1, \infty}} \lesssim \|\phi_0\|_{H^s}^2$$

This is somewhat wasteful if  $s > 0$  but it is tight for  $s = 0$ . Using duality we rewrite the bound (5.1) in the more symmetric form

$$\left| \int Q_{12}(C, \phi) \psi dx dt \right| \lesssim \|\phi\|_{X^{\sigma, s}} \|\psi\|_{X^{\sigma, -s}} \|\phi_0\|_{H^s}^2$$

Integrating by parts it is easy to see that the null form  $Q_{12}$  can be placed on any two of the factors  $C, \phi, \psi$ . We use the standard Littlewood-Paley trichotomy.

**A. High-low interactions.** Here we only need Bernstein's inequality to write for  $k > j$

$$\begin{aligned} \left| \int Q_{12}(C_k, \phi_j) \psi_k dx dt \right| &\lesssim 2^{k+j} \|C_k\|_{L^1} \|\phi_j\|_{L^\infty} \|\psi_k\|_{L^\infty} \\ &\lesssim 2^{2(j-k)} 2^{-sk} \|u_0\|_{B_s^{1,\infty}} \|\phi_j\|_{L^\infty L^2} \|\psi_k\|_{L^\infty L^2} \\ &\lesssim 2^{2(j-k)} 2^{-sj} \|\phi_0\|_{H^s}^2 \|\phi_j\|_{X^{\sigma,s}} \|\psi_k\|_{X^{\sigma,-s}} \end{aligned}$$

The factor  $2^{-sj}$  is not needed. The summation with respect to  $k$  and  $j$  is ensured by the off-diagonal decay.

**B. High-high interactions.** This case is equivalent to the one above if  $s = 0$  and better if  $s > 0$ .

**C. Low-high interactions.** In this case it suffices to prove the estimate

$$\left| \int Q_{12}(C_{<k}, \phi_k) \psi_k dx dt \right| \lesssim \|\phi_k\|_{X_k^\sigma} \|\psi_k\|_{X_k^\sigma} \|u_0\|_{L^1} \quad (5.2)$$

for some choice of  $\sigma$ . This choice is not important due to the nesting property of the  $X_k^\sigma$  spaces. It is easiest to work with  $\sigma = \frac{1}{2}$ .

By scaling we can take  $k = 0$ . By translation invariance we can take  $u_0 = \delta_0$ . Then  $C$  is radial, and

$$\nabla C_{<0}(x) = xa(|x|, t), \quad |a(r, t)| \lesssim (1 + r + t^{\frac{1}{2}})^{-2}$$

Hence

$$Q_{12}(C_{<0}, \phi_0) = a(|x|, t) \nabla \phi_0$$

which shows that

$$\|Q_{12}(C_{<0}, \phi_0)\|_{X_0^{\frac{1}{2},*}} \lesssim \|\phi_0\|_{X_0^{\frac{1}{2}}}$$

Thus (5.2) follows. □

## 6. BILINEAR ESTIMATES

We define the temporal frequency localization operator  $Q_N^0$  to be the Fourier multiplier with symbol  $\psi_N(\tau)$  and the modulation localization operator  $Q_N$  to be the Fourier multiplier with symbol  $\psi_N(\tau - \xi^2)$ . Here  $\psi_N$  is the same bump function we used in (4.8) to define Littlewood-Paley projections.

In the rest of the paper,  $Q_N$  will be applied to single functions whereas  $Q_N^0$  will be applied to bilinear expressions.

In the following, we will sometimes use the notation  $U^2$ , which stands for both  $U_\Delta^2$  and  $|D|^{-\frac{1}{2}} U_\mathbf{e}^2$ . The same convention holds for  $V^2$ .

**6.1. Pointwise bilinear estimates.** These are needed for the case of balanced frequency interactions. Let  $I_\lambda$  denote the frequency annulus  $\{\xi \in \mathbb{R}^2 : \lambda/2 \leq |\xi| \leq 2\lambda\}$ . Our first result is

**Lemma 6.1.** *Let  $\mu, \nu, \lambda$  be dyadic frequencies satisfying  $\mu \ll \lambda$  and  $\nu \lesssim \mu\lambda$ . Let  $\phi_\lambda, \psi_\lambda$  be functions with frequency support contained in  $I_\lambda$ . Then*

$$\|P_\mu Q_\nu^0(\bar{\phi}_\lambda \psi_\lambda)\|_{L^\infty} \lesssim \frac{\mu\nu}{\lambda} \|\phi_\lambda\|_{V^{2,\#}} \|\psi_\lambda\|_{V^{2,\#}} \quad (6.1)$$

*Proof.* We first dispense with the high modulations in the inputs. If both are high ( $\gtrsim \mu\lambda$ ) then by Bernstein we have

$$\|P_\mu Q_\nu^0(\overline{Q_{\gtrsim \mu\lambda} \phi_\lambda} Q_{\gtrsim \mu\lambda} \psi_\lambda)\|_{L^\infty} \lesssim \nu \mu^2 \|(\overline{Q_{\gtrsim \mu\lambda} \phi_\lambda} Q_{\gtrsim \mu\lambda} \psi_\lambda)\|_{L^1} \lesssim \frac{\mu\nu}{\lambda} \|\phi_\lambda\|_{V_\Delta^2} \|\psi_\lambda\|_{V_\Delta^2}$$

If one is high and one is low then we decompose the low modulation factor with respect to small angles and use the lateral energy:

$$\|P_\mu Q_\nu^0(\overline{P_e \phi_\lambda} Q_{\gtrsim \mu\lambda} \psi_\lambda)\|_{L^\infty} \lesssim \nu \mu^{\frac{3}{2}} \|(\overline{P_e \phi_\lambda} Q_{\gtrsim \mu\lambda} \psi_\lambda)\|_{L_e^{2,1}} \lesssim \frac{\mu\nu}{\lambda} \|\phi_\lambda\|_{|D|^{-\frac{1}{2}} V_e^2} \|\psi_\lambda\|_{V_\Delta^2}$$

Finally if both factors are low modulations then we decompose with respect to small angles and compute

$$\|P_\mu Q_\nu^0(\overline{P_e \phi_\lambda} P_e \psi_\lambda)\|_{L^\infty} \lesssim \nu \mu \|(\overline{P_e \phi_\lambda} P_e \psi_\lambda)\|_{L_e^{\infty,1}} \lesssim \frac{\mu\nu}{\lambda} \|P_e \phi_\lambda\|_{|D|^{-\frac{1}{2}} V_e^2} \|P_e \psi_\lambda\|_{|D|^{-\frac{1}{2}} V_e^2}$$

□

A slight sharpening of the above result is as follows:

**Lemma 6.2.** *Let  $\mu, \lambda$  be dyadic frequencies satisfying  $\mu \ll \lambda$ . Let  $\phi_\lambda, \psi_\lambda$  be functions with frequency support contained in  $I_\lambda$ . Then*

$$\|P_\mu H^{-1}(\bar{\phi}_\lambda \partial_x \psi_\lambda)\|_{L^\infty} \lesssim \mu \|\phi_\lambda\|_{V^{2,\#}} \|\psi_\lambda\|_{V^{2,\#}} \quad (6.2)$$

*Proof.* While using Bernstein's inequality as in the previous proof leads to a logarithmic divergence and is no longer immediately useful, we can instead use kernel bounds for  $P_\mu H^{-1}$  with the same effect. The kernel  $K_\mu$  of  $P_\mu H^{-1}$  satisfies

$$|K_\mu(t, x)| \lesssim \mu^2 (1 + \mu|x|)^{-N} (1 + \mu^2|t|)^{-N}$$

Then one can repeat the three cases in the previous proof, but using the kernel bounds instead of Bernstein's inequality. □

**6.2.  $L^2$  bilinear estimates for free solutions.** We introduce an improved bilinear Strichartz estimate that is a slight generalization of that first shown in [3, Lemma 111].

**Lemma 6.3** (Improved bilinear Strichartz). *Let  $u(x, t) = e^{it\Delta} u_0(x), v(x, t) = e^{it\Delta} v_0(x)$ , where  $u_0, v_0 \in L^2(\mathbb{R}^2)$ . Let  $\Omega_1$  denote the support of  $\hat{u}_0(\xi_1)$ ,  $\Omega_2$  the support of  $\hat{v}_0(\xi_2)$ , and set  $\Omega = \Omega_1 \times \Omega_2$ . Assume that  $\Omega_1$  and  $\Omega_2$  are open and separated by some positive distance. Then*

$$\|u\bar{v}\|_{L_{t,x}^2} \lesssim \left( \frac{\sup_{\xi,\tau} \int_{\tau=|\xi_1|^2-|\xi_2|^2}^{\xi=\xi_1-\xi_2} \chi_\Omega(\xi_1, \xi_2) d\mathcal{H}^1(\xi_1, \xi_2)}{\text{dist}(\Omega_1, \Omega_2)} \right)^{1/2} \cdot \|u_0\|_{L^2} \|v_0\|_{L^2} \quad (6.3)$$

where  $d\mathcal{H}^1$  denotes 1-dimensional Hausdorff measure (on  $\mathbb{R}^4$ ) and  $\chi_\Omega(\xi_1, \xi_2)$  the characteristic function of  $\Omega$ .

*Proof.* To control  $\|u\bar{v}\|_{L^2_{t,x}}$ , we are led by duality to estimating

$$\int_{\xi_1, \xi_2} g(\xi_1 - \xi_2, |\xi_1|^2 - |\xi_2|^2) \hat{u}_0(\xi_1) \bar{\hat{v}}_0(\xi_2) d\xi_1 d\xi_2$$

We apply Cauchy-Schwarz and reduce the problem to bounding

$$G := \int_{(\xi_1, \xi_2) \in \Omega} |g(\xi_1 - \xi_2, |\xi_1|^2 - |\xi_2|^2)|^2 d\xi_1 d\xi_2$$

Let  $f : \mathbb{R}^4 \rightarrow \mathbb{R}^3$  be given by  $\mathbb{R}^2 \times \mathbb{R}^2 \ni (\xi_1, \xi_2) \mapsto (\xi_1 - \xi_2, |\xi_1|^2 - |\xi_2|^2) =: (\xi, \tau) \in \mathbb{R}^2 \times \mathbb{R}$ . The differential corresponding to this change of coordinates is

$$df = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 2\xi_1^{(1)} & 2\xi_1^{(2)} & -2\xi_2^{(1)} & -2\xi_2^{(2)} \end{bmatrix}$$

The size  $|J_3 f|$  of the 3-dimensional Jacobian of  $f$  is defined to be the square root of the sum of the squares of the determinants of the  $3 \times 3$  minors of the differential  $df$ :

$$|J_3 f| := 2\sqrt{2} \left( (\xi_2^{(2)} - \xi_1^{(2)})^2 + (\xi_2^{(1)} - \xi_1^{(1)})^2 + (\xi_2^{(2)} - \xi_1^{(2)})^2 + (\xi_1^{(1)} - \xi_1^{(1)})^2 \right)^{1/2}$$

Hence

$$|J_3 f| = C|\xi_2 - \xi_1| \geq C \operatorname{dist}(\Omega_1, \Omega_2) \quad (6.4)$$

By the coarea formula (see [7, §3]),

$$\begin{aligned} G &= \int_{(\xi_1, \xi_2) \in \Omega} |g(\xi_1 - \xi_2, |\xi_1|^2 - |\xi_2|^2)|^2 d\xi_1 d\xi_2 \\ &= \int_{\xi, \tau} \int_{\substack{(\xi_1, \xi_2) \in \Omega: \\ \xi = \xi_1 - \xi_2 \\ \tau = |\xi_1|^2 - |\xi_2|^2}} |g(\xi_1 - \xi_2, |\xi_1|^2 - |\xi_2|^2)|^2 |J_3 f|^{-1}(\xi_1, \xi_2) d\mathcal{H}^1(\xi_1, \xi_2) d\xi d\tau \\ &\leq \int_{\xi, \tau} |g(\xi, \tau)|^2 \int_{\substack{(\xi_1, \xi_2) \in \Omega: \\ \xi = \xi_1 - \xi_2 \\ \tau = |\xi_1|^2 - |\xi_2|^2}} |J_3 f|^{-1}(\xi_1, \xi_2) d\mathcal{H}^1(\xi_1, \xi_2) d\xi d\tau \\ &\leq \int_{\xi, \tau} |g(\xi, \tau)|^2 d\xi d\tau \cdot \sup_{\xi, \tau} \int_{\substack{(\xi_1, \xi_2) \in \Omega: \\ \xi = \xi_1 - \xi_2 \\ \tau = |\xi_1|^2 - |\xi_2|^2}} |J_3 f|^{-1}(\xi_1, \xi_2) d\mathcal{H}^1(\xi_1, \xi_2) \end{aligned} \quad (6.5)$$

In view of (6.4), the right hand side of (6.5) is bounded (up to a constant) by

$$\|g\|_{L^2}^2 \cdot \operatorname{dist}(\Omega_1, \Omega_2)^{-1} \cdot \sup_{\xi, \tau} \int_{\substack{\xi = \xi_1 - \xi_2 \\ \tau = |\xi_1|^2 - |\xi_2|^2}} \chi_{\Omega}(\xi_1, \xi_2) d\mathcal{H}^1(\xi_1, \xi_2)$$

□

A straightforward application of Lemma 6.3 yields

**Corollary 6.4** (Bourgain's improved bilinear Strichartz estimate [3]). *a) Let  $\mu, \lambda$  be dyadic frequencies,  $\mu \ll \lambda$ . Let  $\phi_\mu, \psi_\lambda$  denote free waves respectively localized in frequency to  $I_\mu$  and  $I_\lambda$ . Then*

$$\|\bar{\phi}_\mu \psi_\lambda\|_{L^2} \lesssim \frac{\mu^{1/2}}{\lambda^{1/2}} \|\phi_\mu(0)\|_{L^2_x} \|\psi_\lambda(0)\|_{L^2_x} \quad (6.6)$$

b) If either  $\phi_\mu$  or  $\psi_\lambda$  is further frequency localized to a box of size  $\alpha \times \alpha$ , then we have the better estimate

$$\|\bar{\phi}_\mu \psi_\lambda\|_{L^2} \lesssim \frac{\alpha^{1/2}}{\lambda^{1/2}} \|\phi_\mu(0)\|_{L_x^2} \|\psi_\lambda(0)\|_{L_x^2} \quad (6.7)$$

As a corollary of the proof of Lemma 6.3, we obtain the following.

**Corollary 6.5.** *Let  $u(x, t) = e^{it\Delta} u_0(x)$ ,  $v(x, s) = e^{is\Delta} v_0(x)$ , where  $u_0, v_0 \in L^2(\mathbb{R}^2)$ . Let  $\Omega_1$  denote the support of  $\hat{u}_0(\xi_1)$ ,  $\Omega_2$  the support of  $\hat{v}_0(\xi_2)$ . Assume that for all  $\xi_1 \in \Omega_1$  and  $\xi_2 \in \Omega_2$  we have*

$$|\xi_1 \wedge \xi_2| \sim \beta$$

Then

$$\|u\bar{v}\|_{L_{s,t,x}^2} \lesssim \beta^{-1/2} \|u_0\|_{L^2} \|v_0\|_{L^2} \quad (6.8)$$

*Proof.* As in the proof of Lemma 6.3, we use a duality argument. The key is to bound

$$\int_{(\xi_1, \xi_2) \in \Omega} |g(\xi_1 - \xi_2, |\xi_1|^2, |\xi_2|^2)| d\xi_1 d\xi_2$$

in  $L^2$ . In this setting, the proof is simpler because the change of variables  $f$  is given by  $\mathbb{R}^2 \times \mathbb{R}^2 \ni (\xi_1, \xi_2) \mapsto (\xi_1 - \xi_2, |\xi_1|^2, |\xi_2|^2) \in \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$  so that  $f : \mathbb{R}^4 \rightarrow \mathbb{R}^4$  and  $|df| \sim |\xi_1 \wedge \xi_1|$ .  $\square$

In order to achieve a gain at matched frequencies, we localize the output in both frequency and modulation, seeking to bound  $P_\mu Q_\nu(\bar{\phi}_\lambda \psi_\lambda)$  in  $L^2$ . That Lemma 6.3 may be used efficiently, we introduce an adapted frequency-space decomposition of annuli  $I_\lambda \subset \mathbb{R}^2$  that depends upon both the output frequency and modulation cutoff scales  $\mu$  and  $\nu$ .

**Definition 6.6** (Frequency decomposition). Suppose  $\mu, \nu, \lambda$  are dyadic frequencies satisfying  $\mu \ll \lambda$  and  $\nu \leq \mu\lambda$ . We define a partition of  $I_\lambda$  into curved boxes as follows. First, partition  $I_\lambda$  into  $\lambda^2/\nu$  annuli of equal thickness. Next, uniformly partition the annuli into  $\lambda/\mu$  sectors of equal angle. The resulting set of curved boxes we call  $\mathcal{Q} = \mathcal{Q}(\mu, \nu, \lambda)$ .

The curved sides of the boxes in  $\mathcal{Q}$  have length  $\sim \mu$ , whereas the straight sides of the boxes have length  $\sim \nu/\lambda$ . By adapting a suitable partition of unity to the decomposition, we have

$$f = \sum_{\substack{\mu \ll \lambda \\ \nu \leq \mu\lambda}} \sum_{R \in \mathcal{Q}(\mu, \nu, \lambda)} P_R f$$

Note that we may extend  $\mathcal{Q}(\mu, \nu, \lambda)$  to all smaller dyadic scales  $\lambda' < \lambda$  in the following way: take the partition  $\mathcal{Q}(\mu, \nu, \lambda')$  and cut the annuli into  $\lambda/\lambda'$  smaller annuli of equal thickness. In this way we can impose a finer scale on lower frequencies.

**Corollary 6.7.** *Let  $\mu, \nu, \lambda$  be dyadic frequencies satisfying  $\mu \lesssim \lambda$  and  $\nu \lesssim \mu\lambda$ . Let  $\phi_\lambda, \psi_\lambda$  be free waves with frequency support contained in  $I_\lambda$ . Then*

$$\|P_\mu Q_\nu^0(\bar{\phi}_\lambda \psi_\lambda)\|_{L^2} \lesssim \frac{\nu^{1/2}}{(\mu\lambda)^{1/2}} \|\phi_\lambda\|_{L_x^2} \|\psi_\lambda\|_{L_x^2} \quad (6.9)$$

*Proof.* The frequency restriction  $P_\mu$  applied to  $P_R \bar{\phi}_\lambda P_{R'} \psi_\lambda$  restricts us to looking at the subcollection of boxes  $R, R' \in \mathcal{Q}$  separated by a distance  $\sim \mu$ . This subcollection is further restricted by the temporal frequency multiplier  $Q_\nu^0$ . Let  $\xi_1 \in R, \xi_2 \in R'$ . The modulation  $\tau$  of the product  $P_R \bar{\phi}_\lambda P_{R'} \psi_\lambda$  is given by

$$|\xi_1|^2 - |\xi_2|^2 = (\xi_1 - \xi_2) \cdot (\xi_1 + \xi_2)$$



Because we apply  $P_\mu$ ,  $|\xi_1 - \xi_2| \sim \mu$ , and therefore necessarily  $\tau$  lies in the range  $|\tau| \lesssim \mu\lambda$ . We write  $|\tau| \sim \mu\lambda \cos \theta$ , where  $\theta$  is the angle between  $\xi_1 - \xi_2$  and  $\xi_1 + \xi_2$ . Applying  $Q_\nu$  restricts  $\tau$  so that  $|\tau| \sim \nu$  and in particular  $|\cos \theta| \sim \nu/(\mu\lambda)$ .

These restrictions motivate defining the set of interacting pairs of boxes  $\mathcal{P} = \mathcal{P}(\mu, \nu, \lambda)$  as the collection of all pairs  $(R, R') \in \mathcal{Q} \times \mathcal{Q}$  ( $\mathcal{Q} = \mathcal{Q}(\mu, \nu, \lambda)$ ) for which all  $(\xi_1, \xi_2) \in R \times R'$  satisfy  $||\xi_1|^2 - |\xi_2|^2| \sim \mu$  and  $|\xi_1 + \xi_2| \sim \nu$ .

Note that, for  $R \in \mathcal{Q}(\mu, \nu, \lambda)$  fixed, the number  $p$  of interacting pairs of boxes  $P \in \mathcal{P}(\mu, \nu, \lambda)$  containing  $R$  is  $O(1)$  uniformly in  $\mu, \nu, \lambda$ . This is a consequence of the restrictions  $|\cos \theta| \sim \nu/(\mu\lambda)$  and  $|\xi| \sim \mu$ : they jointly enforce at most  $O(1)$  translations of a distance  $\sim \nu/\lambda$ , which is precisely the scale of the short sides of the boxes.

It remains only to show that for  $(R, R') \in \mathcal{P}$  we have

$$\sup_{\xi, \tau} \int_{\substack{\xi = \xi_1 - \xi_2 \\ \tau = |\xi_1|^2 - |\xi_2|^2}} \chi_R(\xi_1) \chi_{R'}(\xi_2) d\mathcal{H}^1(\xi_1, \xi_2) \lesssim \frac{\nu}{\lambda} \quad (6.10)$$

Fix  $\xi \in \mathbb{R}^2, \tau \in \mathbb{R}, \xi \neq 0$ , and consider the constraint equations

$$\begin{cases} \xi &= \xi_1 - \xi_2 \\ \tau &= |\xi_1|^2 - |\xi_2|^2 \end{cases} \quad (6.11)$$

These determine a line in  $\mathbb{R}^2$ :

$$\tau = (\xi_1 - \xi_2) \cdot (\xi_1 + \xi_2) = -\xi \cdot (\xi - 2\xi_1)$$

Suppose this line intersects  $R$ . The angle  $\rho$  that it forms with the long side length of  $R$  satisfies  $|\cos \rho| \sim \nu/(\mu\lambda)$  due to the modulation constraint (note that at the scale of these boxes, the effects of curvature can be neglected). Since the long side of  $R$  has length  $\sim \mu$  and the short side length  $\sim \nu/\lambda$ , it follows that the total intersection length is  $O(\nu/\lambda)$ .  $\square$

**6.3. Extensions.** Now we extend the bilinear estimates to  $U_\Delta^2$  and  $|D|^{-\frac{1}{2}} U_e^2$  functions; since these extensions are valid for both spaces, we simplify the notation to  $U^2$ .

Our first application of this proposition is in observing that (6.6) of Corollary 6.4 extends to  $U^2$  functions. This follows easily from the atomic decomposition.

**Corollary 6.8.** *Let  $\phi_\mu, \psi_\lambda \in U^2$  be respectively localized in frequency to  $I_\mu$  and  $I_\lambda$ ,  $\mu \ll \lambda$ . Then*

$$\|\bar{\phi}_\mu \psi_\lambda\|_{L^2} \lesssim \frac{\mu^{1/2}}{\lambda^{1/2}} \|\phi_\mu\|_{U^2} \|\psi_\lambda\|_{U^2} \quad (6.12)$$

We may similarly conclude the following.

**Corollary 6.9.** *Let  $\phi_1, \phi_2 \in U^2$  be respectively localized in frequency to  $\Omega_1$  and  $\Omega_2$ , where  $\Omega_1, \Omega_2 \subset I_\lambda$ . Assume that for all  $\xi_1 \in \Omega_1$  and  $\xi_2 \in \Omega_2$  we have*

$$|\xi_1 \wedge \xi_2| \sim \beta.$$

*Then*

$$\|\phi_1 \bar{\phi}_2\|_{L^2_{s,t,x}} \lesssim \beta^{-1/2} \|\phi_1\|_{U^2} \|\phi_2\|_{U^2} \quad (6.13)$$

**Lemma 6.10.** *Let  $Q_1, Q_2 \in \{Q_{\leq \nu_1}, Q_{\nu_2}, Q_{\geq \nu_3}, 1 : \nu_1, \nu_2, \nu_3 \text{ dyadic}\}$ . Let  $\phi_\mu, \phi_\lambda \in U^2$  have respective frequency supports contained in  $\alpha$  boxes lying in  $I_\mu$  and  $I_\lambda$ , where  $\mu \ll \lambda$ . Then*

$$\|Q_1 \phi_\mu \cdot Q_2 \phi_\lambda\|_{L^2} \lesssim \frac{\alpha^{1/2}}{\lambda^{1/2}} \|\phi_\mu\|_{U^2} \|\phi_\lambda\|_{U^2} \quad (6.14)$$

*Proof.* First consider the case where  $Q_1 = 1$  and  $Q_2$  is of the form  $Q_{\leq \nu}$ . As  $Q_2$  is a Fourier multiplier with (Schwartz) symbol

$$b(\xi, \tau) := \chi((\tau - |\xi|^2)/\nu)$$

we have

$$Q_2 \phi_\lambda(x, t) = (\tilde{b} * \phi_\lambda)(x, t) = \int \tilde{b}(y, s) \phi_\lambda(x - y, t - s) dy ds$$

and so it follows that the left hand side of (6.14) admits the representation

$$\|\phi_\mu(x, t) \int \tilde{b}(y, s) \phi_\lambda(x - y, t - s) dy ds\|_{L^2_{x,t}}$$

Suppose we freeze  $y, s$  and consider

$$\|\phi_\mu(x, t) \tilde{\phi}_\lambda(x - y, t - s)\|_{L^2_{x,t}}$$

By replacing  $\phi_\mu$  and the translated  $\phi_\lambda$  with atoms, we obtain by Lemma 6.3 and the fact that the  $U^2$  spaces are translation invariant that

$$\|\phi_\mu(x, t) \tilde{\phi}_\lambda(x - y, t - s)\|_{L^2_{x,t}} \lesssim \frac{\alpha^{1/2}}{\lambda^{1/2}} \|\phi_\mu\|_{U^2} \|\phi_\lambda\|_{U^2}$$

Since  $\tilde{b}(x, t)$  is integrable with bound independent of  $\nu$ , (6.14) follows in this special case.

This argument clearly generalizes to  $Q_1, Q_2 \in \{Q_{\leq \nu_1}, Q_{\nu_2}, 1 : \nu_1, \nu_2 \text{ dyadic}\}$ . In order to accommodate  $Q_{\geq \nu}$ , we apply the above argument to  $1 - Q_{\geq \nu}$  and use the triangle inequality.  $\square$

**Lemma 6.11.** *Let  $\phi_\mu, \phi_\lambda \in \ell^2 V^{2,\sharp}$  be respectively frequency localized to dyadic scales  $\mu, \lambda$ . Then*

$$\|\phi_\mu \phi_\lambda\|_{L^2} \lesssim \frac{\mu^{1/2}}{\lambda^{1/2}} (\log \mu)^2 \|\phi_\mu\|_{\ell^2 V^{2,\sharp}} \|\phi_\lambda\|_{V^{2,\sharp}} \quad (6.15)$$

*b) If either  $\phi_\mu$  or  $\psi_\lambda$  is further frequency localized to space-time boxes of respective size  $\alpha \times \alpha \times \alpha \mu$  or  $\alpha \times \alpha \times \alpha \lambda$ , then we have the better estimate*

$$\|\phi_\mu \phi_\lambda\|_{L^2} \lesssim \frac{\alpha^{1/2}}{\lambda^{1/2}} (\log \mu)^2 \|\phi_\mu\|_{\ell^2 V^{2,\sharp}} \|\phi_\lambda\|_{V^{2,\sharp}} \quad (6.16)$$

*Proof.* We first easily dispense with the high modulations ( $\gtrsim \mu\lambda$ ) in  $\phi_\lambda$  by using the trivial pointwise bound for  $\phi_\mu$ . Once  $\phi_\lambda$  is restricted to small modulations we localize it to small angles. On the other hand, for  $\phi_\mu$  we take advantage of the  $\ell^2$  structure to localize the estimate to a cube of size  $\mu \times \mu \times 1$ , using a smooth cutoff  $\chi_\mu$ . Thus it remains to estimate the expression  $\chi_\mu \phi_\mu P_{\mathbf{e}} \phi_\lambda$ .

Our starting point is the estimate (6.12) from Corollary 6.8, which yields the correct bound for  $\phi_\lambda$  in  $|D|^{-\frac{1}{2}} U_{\mathbf{e}}^2$  and  $\phi_\mu$  in  $U_{\Delta}^2$ . Next we put  $\phi_\lambda$  in  $|D|^{-\frac{1}{2}} U_{\mathbf{e}}^p$ , which embeds into  $\lambda^{-1/2} L_{\mathbf{e}}^{\infty,2}$ , and  $\phi_\mu$  in  $V_{\Delta}^p$ , which has the property that  $\chi P_{\mu} V_{\Delta}^p \subset \chi \mu^2 L^{\infty} \subset \mu^{\frac{5}{2}} L_{\mathbf{e}}^{2,\infty}$ . Therefore by Lemma 4.8

$$\|P_{\mathbf{e}} \phi_\lambda \chi_\mu \phi_\mu\|_{L^2} \lesssim \frac{\mu^{1/2}}{\lambda^{1/2}} (\log \mu^{N-1/2} + 1)^2 \|\phi_\lambda\|_{V_{\mathbf{e}}^2} \|\phi_\mu\|_{V_{\Delta}^2}$$

$\square$

**Lemma 6.12.** *Let  $\mu, \nu, \lambda$  be dyadic frequencies satisfying  $\mu \ll \lambda$  and  $\mu^2 \lesssim \nu \lesssim \mu\lambda$ . Let  $\phi_\lambda, \psi_\lambda$  be waves with frequency support contained in  $I_\lambda$ . Then*

$$\|P_{\mu} Q_{\nu}^0 (\overline{P_{\mathbf{e}} Q_{\lesssim \mu\lambda} \phi_\lambda} \cdot P_{\mathbf{e}} Q_{\lesssim \mu\lambda} \psi_\lambda)\|_{L^2} \lesssim \frac{\nu^{1/2}}{(\mu\lambda)^{1/2}} \|\phi_\lambda\|_{|D|^{-\frac{1}{2}} U_{\mathbf{e}}^2} \|\psi_\lambda\|_{|D|^{-\frac{1}{2}} U_{\mathbf{e}}^2} \quad (6.17)$$

where  $Q_{\nu}^0$  is replaced by  $Q_{\lesssim \mu^2}^0$  if  $\nu \leq \mu^2$ .

*Proof.* Notice that at frequency  $\lambda$ , the characteristic surface has slope  $\lambda$ . Once we apply the modulation localization  $Q_{\lesssim \mu \lambda}$ ,  $\phi_\lambda$  and  $\psi_\lambda$  will be localized horizontally to boxes of size  $\mu \times \mu$ .

Using the square summability in Lemma 4.11 with respect to vertical rectangles of size  $\mu \times \nu$  in the  $\mathbf{e}^\perp$  plane, and then the modulation localization, the problem reduces to the case of free waves which are localized in horizontally and vertically separated cubes of size  $\mu \times \mu \times \nu$ . Then we can drop both  $P_\mu$  and  $Q_\nu^0$ , and apply the localized version of Lemma 6.7 extended to  $U_{\mathbf{e}}^2$  atoms.

□

## 7. ESTIMATES ON $A, B$

In this section we consider the system (3.4) for  $A_1$  and  $A_2$  under the assumption that  $\phi$  is small in the space  $X^s$ . Unfortunately, it does not seem possible to directly obtain estimates for  $A_1, A_2$  which suffice in order to close the rest of the argument. For this reason, we express  $A_1, A_2$  in terms of the milder variable  $B_1, B_2$ , which we can estimate in better norms and treat perturbatively. Thus we end up solving the system (3.5) perturbatively.

We begin with estimates for the first two components of  $A_1, A_2$  in (3.4).

**Lemma 7.1** (Linear flow estimate). *Assume that  $\phi_0 \in H^s$ . Then we have*

$$\langle D \rangle^s H^{-1} A_x(0) \in L_{x,t}^4[0, 1] \cap L_{\mathbf{e}}^{2,6}[0, 1] \quad (7.1)$$

*Proof.* We recall that  $A_1(0) = \frac{1}{2} \Delta^{-1} \partial_2 |\phi_0|^2$  and  $A_2(0) = -\frac{1}{2} \Delta^{-1} \partial_1 |\phi_0|^2$ . We split the data into low and high frequency components. For the high frequency part we have the multiplicative estimate

$$\| \langle D \rangle^s P_{>0} A_x(0) \|_{L^2} = \| \langle D \rangle^s \Delta^{-1} \partial_x |\phi_0|^2 \|_{L^2} \lesssim \| \phi_0 \|_{H^s}^2$$

By parabolic regularity this gives

$$\| \langle D \rangle^s P_{>0} H^{-1} A_x(0) \|_{L_t^2 \dot{H}_x^1 \cap L_x^2 L_t^\infty} \lesssim \| \phi_0 \|_{H^s}^2$$

and the conclusion follows by Sobolev embeddings.

For the low frequency part we can only use the straightforward estimate

$$\| |u_0|^2 \|_{L^1} \lesssim \| u_0 \|_{H^s}^2$$

We write

$$P_{<0} H^{-1} A_x(0) = K(t) * |\phi_0|^2$$

where the kernel  $K(t)$  of  $\Delta^{-1} \partial_x H^{-1} P_{<0}$  is given by

$$K(t) = \mathcal{F}^{-1}(\psi(|\xi|) \frac{\xi_j}{|\xi|^2} e^{-t|\xi|^2})$$

and  $\psi(\xi)$  is the symbol of the Littlewood-Paley projection  $P_{<0}$ . For  $K(t, x)$ , we notice the following two facts:

- $|K(t, x)| \lesssim 1$ . This is because

$$|K(t, x)| \lesssim \int \frac{|\xi_j|}{|\xi|^2} \psi(|\xi|) |\xi| |d\xi| \lesssim \int \psi(r) dr$$

- $K(t, x) \sim \frac{x_j}{|x|^2}$  as  $x \rightarrow \infty$ . In radial coordinates,  $(\xi_1, \xi_2) = (r \cos \theta, r \sin \theta)$ , and we can assume  $\xi_j$  to be  $\xi_1$ . We write

$$\begin{aligned} K(t, x) &= \int_0^{2\pi} \int_0^\infty \frac{\psi(r) r \cos \theta}{r^2} e^{ir(x_1 \cos \theta + x_2 \sin \theta)} e^{-tr^2} r dr d\theta \\ &= \int_0^{2\pi} \int_0^\infty \psi(r) \cos \theta e^{i|x|r(\frac{x_1}{|x|} \cos \theta + \frac{x_2}{|x|} \sin \theta)} e^{-tr^2} dr d\theta \end{aligned}$$

By the method of stationary phase, we get the asymptotic  $\frac{x_j}{|x|^2}$ .

Thus we have the uniform bound

$$|K(t, x)| \lesssim \langle x \rangle^{-1}$$

With this, we conclude that

$$\|P_{<0} H^{-1} A_x(0)\|_{L_x^4} \lesssim \|K(t, x)\|_{L^4} \|\phi_0\|_{L_x^1}^2 \lesssim \|u_0\|_{H^s}^2$$

$$\|P_{<0} H^{-1} A_x(0)\|_{L_e^{2,6}} \lesssim \|K(t, x)\|_{L_e^{2,6}} \|\phi_0\|_{L_x^1}^2 \lesssim \|u_0\|_{H^s}^2$$

and the proof of the lemma is concluded.  $\square$

**Lemma 7.2.** *We have the following bounds for the Littlewood-Paley pieces of  $H^{-1}(\bar{\phi} \partial_x \phi)$ :*

$$\|H^{-1} P_{\lambda_3}(\bar{\phi}_{\lambda_1} \partial_x \phi_{\lambda_2})\|_{\sum_e L_e^{\infty,3}} \lesssim \|\phi_{\lambda_1}\|_{V^{2,\#}} \|\phi_{\lambda_2}\|_{V^{2,\#}}, \quad \lambda_1 \sim \lambda_2 \gg \lambda_3 \quad (7.2)$$

$$\|H^{-1} P_{\lambda_3}(\bar{\phi}_{\lambda_1} \partial_x \phi_{\lambda_2})\|_{H^{-\frac{1}{2}} L_{x,t}^2} \lesssim \|\phi_{\lambda_1}\|_{V^{2,\#}} \|\phi_{\lambda_2}\|_{V^{2,\#}}, \quad \lambda_3 \sim \max\{\lambda_1, \lambda_2\} \quad (7.3)$$

In addition, if  $\lambda_3 \ll \lambda_1 \sim \lambda_2$ , then

$$\|H^{-1} P_{\lambda_3} \left( \bar{\phi}_{\lambda_1} \partial_x \phi_{\lambda_2} - Q_{\lesssim \lambda_1 \lambda_3}^0(\overline{Q_{\lesssim \lambda_1 \lambda_3} \bar{\phi}_{\lambda_1}} \partial_x Q_{\lesssim \lambda_1 \lambda_3} \phi_{\lambda_2}) \right)\|_{H^{-\frac{1}{2}} L_{x,t}^2} \lesssim \|\phi_{\lambda_1}\|_{V^{2,\#}} \|\phi_{\lambda_2}\|_{V^{2,\#}} \quad (7.4)$$

*Proof.* For the first estimate we first dispense with the case when either factor has high modulation. Indeed, assume the first factor has high modulation. Then we have

$$\|Q_{\gtrsim \lambda_1} \bar{\phi}_{\lambda_1} \partial_x \phi_{\lambda_2}\|_{L^{\frac{4}{3}}} \lesssim \|Q_{\gtrsim \lambda_1} \bar{\phi}_{\lambda_1}\|_{L^2} \|\partial_x \phi_{\lambda_2}\|_{L^4} \lesssim \|\phi_{\lambda_1}\|_{V^{2,\#}} \|\phi_{\lambda_2}\|_{V^{2,\#}}$$

and the conclusion follows due to the parabolic Sobolev embedding

$$P_\lambda H^{-1} : L^{\frac{4}{3}} \rightarrow L_e^{\infty,3}$$

Next we assume both factors have small modulations and localize the two factors to small close angular sectors. This is possible since the output frequency is much lower. Then we choose a common admissible direction  $\mathbf{e}$ , and bound both factors in  $L_e^{\infty,2}$ ,

$$\|P_{\mathbf{e}} \bar{\phi}_{\lambda_1} \partial_x P_{\mathbf{e}} \phi_{\lambda_2}\|_{L_e^{\infty,1}} \lesssim \|\phi_{\lambda_1}\|_{V^{2,\#}} \|\phi_{\lambda_2}\|_{V^{2,\#}}$$

Then the conclusion follows due to the parabolic Sobolev embedding

$$P_\lambda H^{-1} : L_e^{\infty,1} \rightarrow L_e^{\infty,3}$$

The second estimate (7.3) is easier. We bound both factors in  $L^4$  and use the parabolic Sobolev embedding

$$\lambda P_\lambda H^{-\frac{1}{2}} : L^2 \rightarrow L^2$$

For the third estimate (7.4) we need to consider two cases for the terms in the difference:

- (i) Both inputs have high modulation. Then we need to estimate

$$H^{-1} P_{\lambda_3}(\overline{Q_{\gg \lambda_1 \lambda_3} \bar{\phi}_{\lambda_1}} \partial_x Q_{\gg \lambda_1 \lambda_3} \phi_{\lambda_2})$$

Placing the two factors in  $L^2$ , we conclude using the bound

$$H^{-\frac{1}{2}}P_{\lambda_3} : L^1 \rightarrow \lambda_3 L^2$$

(ii) One input and the output have high modulation. Then we need to estimate

$$H^{-1}P_{\lambda_3}Q_{\gg_{\lambda_1\lambda_3}}^0(\overline{Q_{\gg_{\lambda_1\lambda_3}}}\phi_{\lambda_1}\partial_x\phi_{\lambda_2})$$

Placing the first factor in  $L^2$  and the second in  $L^\infty L^2$ , we conclude using the bound

$$H^{-\frac{1}{2}}P_{\lambda_3}Q_{\gg_{\lambda_1\lambda_3}}^0 : L^2 L^1 \rightarrow (\lambda_3/\lambda_1)^{\frac{1}{2}} L^2$$

□

**Lemma 7.3.** *Assume that  $\phi$  is small in  $X^s$ . Then the system (3.5) admits a unique solution  $B_1, B_2$ , depending smoothly on  $\phi$ , and with regularity*

$$\langle D \rangle^s B \in H^{-\frac{1}{2}}L_{x,t}^2[0, 1] \quad (7.5)$$

*Remark 7.4.* Notice the following Sobolev embeddings hold:

$$H^{-1}(L_{\mathbf{e}}^{2,6/5}) \hookrightarrow H^{-\frac{1}{2}}L_{t,x}^2 \quad (7.6)$$

$$H^{-\frac{1}{2}}L_{t,x}^2 \hookrightarrow L_{t,x}^4 \quad (7.7)$$

$$H^{-\frac{1}{2}}L_{t,x}^2 \hookrightarrow L_{\mathbf{e}}^{2,6} \quad (7.8)$$

So  $\langle D \rangle^s H^{-1}(A_x|\phi|^2) \in L_{t,x}^4 \cap L_{\mathbf{e}}^{2,6}$ . This is also true for  $\langle D \rangle^s H^{-1}P_{\lambda_3}(\bar{\phi}_{\lambda_1}\partial_x\phi_{\lambda_2})$  when  $\lambda_3 \sim \max(\lambda_1, \lambda_2)$ .

*Proof.* Let us recall the formula (3.5) for  $B$

$$B_1 = H^{-1}((H^{-1}A_2(0) + H^{-1}[\operatorname{Re}(\bar{\phi}\partial_1\phi) + \operatorname{Im}(\bar{\phi}\partial_1\phi)])|\phi|^2) + H^{-1}(B_2|\phi|^2)$$

$$B_2 = H^{-1}((H^{-1}A_1(0) - H^{-1}[\operatorname{Re}(\bar{\phi}\partial_2\phi) + \operatorname{Im}(\bar{\phi}\partial_2\phi)])|\phi|^2) - H^{-1}(B_1|\phi|^2)$$

We use the contraction principle to solve for  $B$  in the space in the lemma. We begin with the first term on the right. By (7.1) we have  $\langle D \rangle^s H^{-1}A_x(0) \in L_{x,t}^4[0, 1]$ . Also by virtue of the Strichartz estimate (4.12), we have  $\langle D \rangle^s \phi \in L_{x,t}^4$ . Then we obtain

$$\langle D \rangle^s H^{-1}(H^{-1}A_x(0)|\phi|^2) \in H^{-1}L^{\frac{4}{3}} \subset H^{-\frac{1}{2}}L_{x,t}^2[0, 1]$$

By the second part of Lemma 7.2 and the embedding  $H^{-\frac{1}{2}}L^2 \subset L^4$ , the same argument applies for the low  $\times$  high  $\rightarrow$  high contribution in the second term in the formula above, as well as for the third term.

For the high  $\times$  high  $\rightarrow$  low contribution we need to use the first part of Lemma 7.2, and so we only have the  $L_{\mathbf{e}}^{\infty,3}$  norm available; however, a redeeming feature is that we obtain  $l^2$  dyadic summation. We also have  $l^2$  dyadic summation in the  $L^4$  bound for  $\langle D \rangle^s \phi$ , and this is still sufficient in view of the embedding

$$P_{\lambda}H^{-\frac{1}{2}} : L_{\mathbf{e}}^{2,\frac{6}{5}} \rightarrow L^2$$

□

## 8. CUBIC TERMS

In this section, we focus on controlling the main cubic terms of the nonlinearity, i.e., (3.8), (3.9), and (3.10). The term  $N_{3,3}$  is controlled using the  $L^4$  Strichartz estimate. Controlling  $N_{3,1}$  and  $N_{3,2}$  requires considerable additional work. Our strategy is as follows. By duality, we can rephrase the estimate for  $N_{3,1}$  as a quadrilinear estimate

$$\left| \int N_{3,1}(\phi^{(1)}, \phi^{(2)}, \phi^{(3)}, \phi^{(4)}) \, dx dt \right| \lesssim \|\phi^{(1)}\|_{X^{\sharp,s}} \|\phi^{(2)}\|_{X^{\sharp,s}} \|\phi^{(3)}\|_{X^{\sharp,s}} \|\phi^{(4)}\|_{X^{-s}}$$

where by a slight abuse of notation we set

$$N_{3,1}(\phi^{(1)}, \phi^{(2)}, \phi^{(3)}, \phi^{(4)}) = H^{-1}(\bar{\phi}^{(1)} \partial_1 \phi^{(2)})(\bar{\phi}^{(3)} \partial_2 \phi^{(4)}) - H^{-1}(\bar{\phi}^{(1)} \partial_2 \phi^{(2)})(\bar{\phi}^{(3)} \partial_1 \phi^{(4)})$$

The same can be done for  $N_{3,2}$ . After one integration by parts, the corresponding quadrilinear form is split into one part which is identical to the one above, and one which is similar but with the complex conjugation on  $\phi^{(4)}$  instead of  $\phi^{(3)}$ . The arguments that follow apply equally to both cases.

Because the four input frequencies  $\xi_j$  satisfy

$$\xi_1 - \xi_2 + \xi_3 - \xi_4 = 0, \tag{8.1}$$

the four input frequencies  $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$  are not completely independent. Two of these are essentially at the same scale,  $\lambda$ , and dominate the remaining ones, whose scales we now call  $\mu_1$  and  $\mu_2$ . We assume that  $\mu_1 \leq \mu_2$ . Examining the balance of frequencies coming from the  $H^s$  norms in the estimate above, we have that the worst case is when the high frequencies cancel and we only can use the low frequency factors. To the above we add a final frequency parameter  $\alpha$  which is the  $H^{-1}$  frequency. We denote by  $N_{3,1}^\alpha$  the expression obtained from  $N_{3,1}$  by replacing  $H^{-1}$  by  $P_\alpha H^{-1}$ .

Relaxing also the  $X^{\sharp,s}$  norms to  $X^s$  and then retaining only the  $l^2 V^{2,\sharp}$  component, we are left with proving the estimate

$$\left| \int N_{3,1}^\alpha(\phi_{\lambda_1}^{(1)}, \phi_{\lambda_2}^{(2)}, \phi_{\lambda_3}^{(3)}, \phi_{\lambda_4}^{(4)}) \, dx dt \right| \lesssim \mu_2^s \|\phi_{\lambda_1}^{(1)}\|_{l^2 V^{2,\sharp}} \|\phi_{\lambda_2}^{(2)}\|_{l^2 V^{2,\sharp}} \|\phi_{\lambda_3}^{(3)}\|_{l^2 V^{2,\sharp}} \|\phi_{\lambda_4}^{(4)}\|_{l^2 V^{2,\sharp}} \tag{8.2}$$

Here the  $\lambda$  summation is produced from the  $l^2$  dyadic summations for the frequency  $\lambda$  factors, while the  $\mu_1$  and  $\mu_2$  summations are satisfactory due to the frequency factor above (applied with a smaller  $s$ ). For  $\alpha$  we must have either  $\alpha \approx \lambda$  or  $\alpha \lesssim \mu_2$ , and so the  $\alpha$  summation also yields at most  $\log \mu_2$  acceptable losses.

We distinguish cases according to whether the highest two input frequencies are balanced or unbalanced, where we say that a pair  $\phi_{\lambda_{2j-1}} \phi_{\lambda_{2j}}$  is *balanced* if  $\lambda_{2j-1} \sim \lambda_{2j}$  and *unbalanced* otherwise. We consider cases in increasing order of difficulty:

**Case 1: Unbalanced case,  $\alpha \approx \lambda$ .** In this case we must have at least one  $\lambda$  frequency factor in each of the two pairs. It suffices to consider the case where the derivatives fall on the high-frequency  $\lambda$  terms. The null structure is not used, and so we need only bound

$$\left| \int P_\lambda H^{-1}(\bar{\phi}_{\mu_1} \partial_1 \phi_\lambda)(\bar{\phi}_{\mu_2} \partial_2 \phi_\lambda) \, dx dt \right| \tag{8.3}$$

To each pair  $\bar{\phi}_{\mu_j} \phi_\lambda$  we apply the bilinear estimate (6.11), obtaining a combined bound of  $(\mu_1 \mu_2)^{1/2} / \lambda$ . The two derivatives in (8.3) are multipliers whose contribution is bounded by  $\lambda^2$ , while  $H^{-1}$  is a multiplier controlled here by  $\lambda^{-2}$ . Therefore (8.3) is bounded by  $O(\mu_1^{1/2} \mu_2^{1/2} / \lambda)$ , and (8.2) follows.

**Case 2: Balanced/unbalanced case,  $\alpha \approx \mu_2 \ll \lambda$ .** Then we need to consider the expression

$$\left| \int P_{\mu_2} H^{-1}(\bar{\phi}_\lambda \partial_1 \phi_\lambda)(\bar{\phi}_{\mu_1} \partial_2 \phi_{\mu_2}) \, dx dt \right| \quad (8.4)$$

We consider three cases:

**Case 2a:** One of the frequency  $\lambda$  factors has high modulation  $\gtrsim \lambda^2$ . We bound that factor in  $L^2$  and all others in  $L^4$ , using

$$\mu P_\mu H^{-1} : L^{\frac{4}{3}} \rightarrow L^2$$

Once this case is dealt with, we can localize both frequency  $\lambda$  factors first to small modulations and then to a small angle.

**Case 2b:** Both  $\phi_{\mu_1}$  and  $\phi_{\mu_2}$  have high modulations. Then we estimate

$$I_{2b} = \int H^{-1}(P_e \bar{\phi}_\lambda \partial_1 P_e \phi_\lambda)(Q_{>\mu_2^2} \bar{\phi}_{\mu_1} \partial_2 Q_{>\mu_2^2} \phi_{\mu_2}) \, dx dt$$

by

$$\begin{aligned} |I_{2b}| &\lesssim \mu_2 \|P_{\mu_2} H^{-1}(P_e \bar{\phi}_\lambda \partial_1 P_e \phi_\lambda)\|_{L^\infty} \|Q_{>\mu_2^2} \phi_{\mu_1}\|_{L^2} \|Q_{>\mu_2^2} \phi_{\mu_2}\|_{L^2} \\ &\lesssim \mu_2^2 \|P_e \bar{\phi}_\lambda \partial_1 P_e \phi_\lambda\|_{L_{\mathbb{e}}^{\infty,1}} \|Q_{>\mu_2^2} \phi_{\mu_1}\|_{L^2} \|Q_{>\mu_2^2} \phi_{\mu_2}\|_{L^2} \\ &\lesssim \|\phi_{\lambda_1}^{(1)}\|_{V^{2,\#}} \|\phi_{\lambda_2}^{(2)}\|_{V^{2,\#}} \|\phi_{\lambda_3}^{(3)}\|_{V^{2,\#}} \|\phi_{\lambda_4}^{(4)}\|_{V^{2,\#}} \end{aligned}$$

**Case 2c:** One of  $\phi_{\mu_1}$  and  $\phi_{\mu_2}$  has low modulation, say  $\phi_{\mu_2}$ . We shift  $H^{-1}$  to the second product, which is localized at frequency  $\mu_2$ ; then we have an expression of the form

$$P_{\mu_2} \bar{H}^{-1}(\phi_{\mu_1} Q_{\leq \mu_2^2} \phi_{\mu_2})$$

The symbol of  $P_{\mu_2} \bar{H}^{-1}$  is smooth on the  $\mu_2^2 \times \mu_2 \times \mu_2$  scale, which is the size of the frequency localization for  $Q_{\leq \mu_2^2} \phi_{\mu_2}$ . Then using Fourier series in both  $\xi$  and  $\tau$ , we can separate variables and replace the above expression by a rapidly convergent sum of the form

$$\sum_j R_j^1 \phi_{\mu_1} R_j^2 Q_{\leq \mu_2^2} \phi_{\mu_2}$$

where  $R_j^1$  has size  $(|\tau| + \mu_2^2)^{-1}$  and dyadic regularity, and  $R_j^2$  is smooth on the scale of the frequency support of  $Q_{\leq \mu_2^2} \phi_{\mu_2}$ . Then we can simply discard the  $R_j^2$  factor and replace  $R_j^1$  by a  $\mu_2^{-2}$  factor. Then in the product estimate

$$I_{2c} = \mu_2^{-2} \int (P_e \bar{\phi}_\lambda \partial_1 P_e \phi_\lambda)(\bar{\phi}_{\mu_1} \partial_2 Q_{\leq \mu_2^2} \phi_{\mu_2}) \, dx dt$$

we regroup terms and use two bilinear  $L^2$  bounds (6.11) to obtain

$$|I_{2c}| \lesssim (\mu_1/\mu_2)^{\frac{1}{2}} (\log \mu_2)^2 \|\phi_{\lambda_1}^{(1)}\|_{V^{2,\#}} \|\phi_{\lambda_2}^{(2)}\|_{V^{2,\#}} \|\phi_{\lambda_3}^{(3)}\|_{V^{2,\#}} \|\phi_{\lambda_4}^{(4)}\|_{V^{2,\#}}$$

**Case 3: Balanced case,  $\alpha \ll \mu_2 \leq \lambda$ .** Then we must have  $\mu_1 \approx \mu_2$ ; we denote both by  $\mu$ . We need to estimate

$$I_3 = \int P_\alpha H^{-1}(\bar{\phi}_\lambda \partial_1 \phi_\lambda)(\bar{\phi}_\mu \partial_2 \phi_\mu) - P_\alpha H^{-1}(\bar{\phi}_\lambda \partial_2 \phi_\lambda)(\bar{\phi}_\mu \partial_1 \phi_\mu) \, dx dt$$

The difficulty in this case is that by using linear and bilinear estimates our losses are in terms of  $\lambda$  and  $\mu$ , while our gains from  $H^{-1}$  are only in terms of  $\alpha$ . This is where our heat gauge is most

useful. We begin by peeling off some easier high modulation cases. The important modulation threshold for  $H^{-1}$  is  $\alpha\lambda$ .

**Case 3a.** *High ( $\gg \alpha\lambda$ ) modulation in  $H^{-1}$ .* This forces at least one comparable modulation in each of the pairs of factors. Thus we need to consider expressions of the form

$$I_{3a} = \int P_\alpha Q_\nu^0 H^{-1}(\bar{\phi}_\lambda \partial_1 Q_{\gtrsim \nu} \phi_\lambda)(\bar{\phi}_\mu \partial_2 Q_{\gtrsim \nu} \phi_\mu) dx dt, \quad \nu \gg \alpha\lambda$$

To bound this we use  $L^2$  for the high modulation factors, energy for the other two and Bernstein at frequency  $\alpha$ . This gives

$$|I_{3a}| \lesssim \frac{\alpha^2 \mu \lambda}{\nu^2} \|\phi_{\lambda_1}^{(1)}\|_{V^{2,\#}} \|\phi_{\lambda_2}^{(2)}\|_{V^{2,\#}} \|\phi_{\lambda_3}^{(3)}\|_{V^{2,\#}} \|\phi_{\lambda_4}^{(4)}\|_{V^{2,\#}}$$

which suffices.

**Case 3b.** *Low ( $\lesssim \alpha\lambda$ ) modulation in  $H^{-1}$  but high ( $\gg \alpha\lambda$ ) modulation in the frequency  $\lambda$  factors.* This forces at least one comparable modulation in each of the pairs of factors. For the frequency  $\mu$  factors the high modulations  $\gtrsim \mu^2$  are easy to treat. Discarding those, we use the relation  $\alpha \ll \mu$  to localize to small angles. Thus we need to consider expressions of the form

$$I_{3b} = \int P_\alpha Q_{\lesssim \alpha\lambda}^0 H^{-1}(\overline{Q_{\gg \alpha\lambda} \phi_\lambda} \partial_1 Q_{\gg \alpha\lambda} \phi_\lambda)(P_{\mathbf{e}} \bar{\phi}_\mu \partial_2 P_{\mathbf{e}} \phi_\mu) dx dt$$

To bound this we use  $L^2$  for the high modulation factors. For the frequency  $\mu$  factors we use lateral energy with respect to the admissible direction  $\mathbf{e}$ . This gives

$$\begin{aligned} |I_{3b}| &\lesssim \lambda \|Q_{\gg \alpha\lambda} \phi_\lambda\|_{L^2} \|Q_{\gg \alpha\lambda} \phi_\lambda\|_{L^2} \|\bar{H}^{-1} P_\alpha (P_{\mathbf{e}} \bar{\phi}_\mu \partial_2 P_{\mathbf{e}} \phi_\mu)\|_{L^\infty} \\ &\lesssim \mu \|\phi_\lambda\|_{V^{2,\#}} \|\phi_\lambda\|_{V^{2,\#}} \|P_{\mathbf{e}} \bar{\phi}_\mu P_{\mathbf{e}} \phi_\mu\|_{L_{\mathbf{e}}^{\infty,1}} \\ &\lesssim \|\phi_{\lambda_1}^{(1)}\|_{V^{2,\#}} \|\phi_{\lambda_2}^{(2)}\|_{V^{2,\#}} \|\phi_{\lambda_3}^{(3)}\|_{V^{2,\#}} \|\phi_{\lambda_4}^{(4)}\|_{V^{2,\#}} \end{aligned}$$

which again suffices.

At this point we can restrict ourselves to low ( $\lesssim \alpha\lambda$ ) modulations in both  $H^{-1}$  and the frequency  $\lambda$  factors. The proof branches again into two cases depending on whether  $\mu \ll \lambda$  or  $\mu \approx \lambda$ . These two cases have similarities but also some significant differences. One such difference is that in the latter case we can freely restrict ourselves to low modulations in all four factors; this turns out to be very useful in our argument.

**Case 3c(i).** *Low ( $\lesssim \alpha\lambda$ ) modulations in both  $H^{-1}$  and the frequency  $\lambda$  factors,  $\mu \ll \lambda$ .* Localizing to small angles in both frequency  $\lambda$  factors, we need to consider the expression

$$\begin{aligned} I_{3c} &= \int P_\alpha Q_{\lesssim \alpha\lambda}^0 H^{-1}(\overline{P_{\mathbf{e}} Q_{\lesssim \alpha\lambda} \phi_\lambda} \partial_1 P_{\mathbf{e}} Q_{\lesssim \alpha\lambda} \phi_\lambda)(\bar{\phi}_\mu \partial_2 \phi_\mu) dx dt \\ &\quad - \int P_\alpha Q_{\lesssim \alpha\lambda}^0 H^{-1}(\overline{P_{\mathbf{e}} Q_{\lesssim \alpha\lambda} \phi_\lambda} \partial_2 P_{\mathbf{e}} Q_{\lesssim \alpha\lambda} \phi_\lambda)(\bar{\phi}_\mu \partial_1 \phi_\mu) dx dt \end{aligned}$$

We will prove that

$$|I_{3c}| \lesssim (\log \mu)^4 \|\phi_{\lambda_1}^{(1)}\|_{V^{2,\#}} \|\phi_{\lambda_2}^{(2)}\|_{V^{2,\#}} \|\phi_{\mu}^{(3)}\|_{l^2 V^{2,\#}} \|\phi_{\mu}^{(4)}\|_{l^2 V^{2,\#}} \quad (8.5)$$

in several steps:

**Case 3c(i), Step 1:** *Proof of (8.5) for free waves, no log loss.* We use the orthogonal partitioning  $\mathcal{Q}(\alpha, \nu, \lambda)$  of  $I_\lambda$  and  $I_\mu$ . Let  $R_1, R_2, R_3, R_4$  be boxes belonging to this partition, where  $R_1, R_2$  are



$\alpha$ -separated at frequency  $\lambda$  and  $R_3, R_4$  are  $\alpha$ -separated at frequency  $\mu$ . By the  $L^2$ -orthogonality of the partition, it suffices to estimate

$$\left| \int H^{-1}(\bar{\phi}_{R_1} \partial_1 \phi_{R_2}) \bar{\phi}_{R_3} \partial_2 \phi_{R_4} dx dt - \int H^{-1}(\bar{\phi}_{R_1} \partial_2 \phi_{R_2}) \bar{\phi}_{R_3} \partial_1 \phi_{R_4} dx dt \right| \quad (8.6)$$

We now split into two subcases according to the strength of the null form.

**Subcase I:** Suppose that there is  $\beta \gtrsim \alpha\lambda$  such that  $|\xi_2 \wedge \xi_4| \sim \beta$  for all  $\xi_2 \in R_2$  and  $\xi_4 \in R_4$ . Then  $|\xi_1 \wedge \xi_3| \sim \beta$  for all  $\xi_1 \in R_1$  and  $\xi_3 \in R_3$ . Let  $b(y, s)$  denote the kernel of  $P_\alpha Q_{<\alpha\lambda}^0 H^{-1}$ . Then

$$|b(y, s)| \lesssim \alpha^{-2} (1 + \alpha|y| + \alpha^2|s|)^{-N} \quad (8.7)$$

Our goal is to control

$$\beta \left| \int b(y, s) (\bar{\phi}_{R_1} \phi_{R_2})(x - y, t - s) (\bar{\phi}_{R_3} \phi_{R_4})(x, t) ds dt dx dy \right|$$

This is bounded by

$$\beta \int \|\bar{\phi}_{R_1}(x - y, t - s) \phi_{R_4}(t, x)\|_{L_{t,s,x}^2} \|\phi_{R_2}(x - y, t - s) \bar{\phi}_{R_3}(t, x)\|_{L_{t,s,x}^2} \sup_s |b(y, s)| dy$$

For the  $L^2$  norm we use the bilinear estimate (6.8) twice, uniformly in  $y$ ; this yields a factor of  $\beta^{-1}$ . Also from (8.7) we have

$$\int \sup_s |b(y, s)| dy \lesssim 1$$

Thus the conclusion follows.

**Subcase II:** Suppose again that  $|\xi_2 \wedge \xi_4| \lesssim \alpha\lambda$  for all  $\xi_2 \in R_2$  and  $\xi_4 \in R_4$ , but now  $\mu \ll \lambda$ . Our goal is now to control

$$\alpha\lambda \left| \int b(y, s) (\bar{\phi}_{R_1} \phi_{R_2})(x - y, t - s) (\bar{\phi}_{R_3} \phi_{R_4})(x, t) ds dt dx dy \right|$$

This is bounded by

$$\alpha\lambda \int \|\bar{\phi}_{R_1}(x - y, t - s) \phi_{R_4}(t, x)\|_{L_{t,x}^2} \|\phi_{R_2}(x - y, t - s) \bar{\phi}_{R_3}(t, x)\|_{L_{t,x}^2} |b(y, s)| ds dy$$

and the argument is concluded by applying (6.6) twice.

**Case 3c(i), Step 2:** *Proof of (8.5) for  $U^2$  waves, no log loss.* Precisely, we will show that

$$|I_{3c}| \lesssim \|P_{\mathbf{e}} \phi_\lambda^{(1)}\|_{|D|^{-\frac{1}{2}} U_{\mathbf{e}}^2} \|P_{\mathbf{e}} \phi_\lambda^{(2)}\|_{|D|^{-\frac{1}{2}} U_{\mathbf{e}}^2} \|\phi_\mu^{(3)}\|_{U_\Delta^2} \|\phi_\mu^{(4)}\|_{U_\Delta^2}, \quad \mu \ll \lambda \quad (8.8)$$

For this we try to mimic the arguments for free waves. The symbol of  $P_\alpha H^{-1} Q_{<\alpha\lambda}^0$  is localized in a region of size  $\alpha \times \alpha \times \alpha\lambda$ . We claim we can freely localize each of the four functions to similarly sized frequency regions. In the case of the  $U_\Delta^2$  spaces for frequency  $\mu$  factors, only the frequency localization on the  $\alpha$  scale is used, and the square summability of the  $U_\Delta^2$  norm with respect to  $\alpha$  rectangles is due to Lemma 4.11. In the case of the frequency  $\lambda$  factors we are using the lateral flow in the  $\mathbf{e}$  direction. Thus by Lemma 4.11 we obtain square summability with respect to projectors associated to vertical rectangles of size  $\alpha \times \alpha\lambda$  in  $\mathbf{e}^\perp$  directions. However, the  $Q_{<\alpha\lambda}$  localization of  $\phi^{(1)}$  and  $\phi^{(2)}$  ensures that the above localization in  $\mathbf{e}^\perp$  directions induces also an  $\alpha$  localization in the  $\mathbf{e}$  direction (this is the same as in the proof of Lemma 6.12). Thus we have arrived at the same situation as in (8.6).

From here on, the proof proceeds exactly as in the free case, using the observation that the bilinear  $L^2$  bounds (6.8) and (6.6) extend in a straightforward manner to  $U^2$  functions.

**Case 3c(i), Step 3:** *Proof of (8.5) for  $U^2$   $\lambda$  waves and  $V^2$   $\mu$ -waves, log loss.* Precisely, we will show that

$$|I_{3c}| \lesssim (\log \mu)^2 \|\phi_\lambda^{(1)}\|_{|D|^{-\frac{1}{2}}U_{\mathbf{e}}^2} \|\phi_\lambda^{(2)}\|_{|D|^{-\frac{1}{2}}U_{\mathbf{e}}^2} \|\phi_\mu^{(3)}\|_{l^2V^{2,\sharp}} \|\phi_\mu^{(4)}\|_{l^2V^{2,\sharp}}, \quad \mu \ll \lambda \quad (8.9)$$

The role of the  $l^2$  structure here is to allow localization to the unit time scale. Once this is done, we decompose

$$\phi_\mu^{(3)} = Q_{<\mu^N} \phi_\mu^{(3)} + Q_{>\mu^N} \phi_\mu^{(3)}$$

For the first component, using the unit time localization, we have

$$\|Q_{<\mu^N} \phi_\mu^{(3)}\|_{U_\Delta^2} \lesssim \log \mu \|\phi_\mu^{(3)}\|_{V_\Delta^2}$$

and use (8.8).

For the second component we estimate directly the quadrilinear form by

$$\begin{aligned} |I_{3c}| &\lesssim \|P_\alpha H^{-1}(\overline{P_e Q_{\lesssim \alpha \lambda} \phi_\lambda} \partial_1 P_e Q_{\lesssim \alpha \lambda} \phi_\lambda)\|_{L^\infty} \|Q_{>\mu^N} \phi_\mu^{(3)}\|_{L^2} \|\partial_x \phi_\mu^{(4)}\|_{L^2} \\ &\lesssim \alpha \|\overline{P_e Q_{\lesssim \alpha \lambda} \phi_\lambda} \partial_1 P_e Q_{\lesssim \alpha \lambda} \phi_\lambda\|_{L_{\mathbf{e}}^{\infty,1}} \mu^{-\frac{N}{2}} \|\phi_\mu^{(3)}\|_{V_\Delta^2} \mu \|\phi_\mu^{(4)}\|_{V_\Delta^2} \\ &\lesssim \alpha \mu^{1-\frac{N}{2}} \|\phi_\lambda^{(1)}\|_{|D|^{-\frac{1}{2}}U_{\mathbf{e}}^2} \|\phi_\lambda^{(2)}\|_{|D|^{-\frac{1}{2}}U_{\mathbf{e}}^2} \|\phi_\mu^{(3)}\|_{l^2V^{2,\sharp}} \|\phi_\mu^{(4)}\|_{l^2V^{2,\sharp}} \end{aligned}$$

**Case 3c(i), Step 4:** *Proof of (8.5), conclusion.* By the interpolation result in Lemma 4.8 the estimate (8.5) follows from the bound (8.9) and the following estimate:

$$|I_{3c}| \lesssim \alpha \mu \|\phi_\lambda^{(1)}\|_{|D|^{-\frac{1}{2}}U_{\mathbf{e}}^p} \|\phi_\lambda^{(2)}\|_{|D|^{-\frac{1}{2}}U_{\mathbf{e}}^p} \|\phi_\mu^{(3)}\|_{l^2V^{2,\sharp}} \|\phi_\mu^{(4)}\|_{l^2V^{2,\sharp}}$$

where  $p > 2$ .

This in turn is obtained as in the immediately preceding computation but without any high modulation localization.

**Case 3c(ii).** *Low ( $\lesssim \alpha \lambda$ ) modulation in both  $H^{-1}$  and the frequency  $\lambda$  factors,  $\mu \approx \lambda$ .* Localizing to small angles in all four frequency  $\lambda$  factors, here we need to consider the expression

$$\begin{aligned} I_{3c} &= \int P_\alpha Q_{\lesssim \alpha \lambda}^0 H^{-1}(\overline{P_e Q_{\lesssim \alpha \lambda} \phi_\lambda} \partial_1 P_e Q_{\lesssim \alpha \lambda} \phi_\lambda)(\overline{P_e Q_{\lesssim \alpha \lambda} \phi_\mu} \partial_2 P_e Q_{\lesssim \alpha \lambda} \phi_\mu) \, dxdt \\ &\quad - \int P_\alpha Q_{\lesssim \alpha \lambda}^0 H^{-1}(\overline{P_e Q_{\lesssim \alpha \lambda} \phi_\lambda} \partial_2 P_e Q_{\lesssim \alpha \lambda} \phi_\lambda)(\overline{P_e Q_{\lesssim \alpha \lambda} \phi_\mu} \partial_1 P_e Q_{\lesssim \alpha \lambda} \phi_\mu) \, dxdt \end{aligned}$$

We will prove that

$$|I_{3c}| \lesssim (\log \mu)^5 \|\phi_\lambda^{(1)}\|_{V^{2,\sharp}} \|\phi_\lambda^{(2)}\|_{V^{2,\sharp}} \|\phi_\mu^{(3)}\|_{l^2V^{2,\sharp}} \|\phi_\mu^{(4)}\|_{l^2V^{2,\sharp}} \quad (8.10)$$

in several steps:

**Case 3c(ii), Step 1:** *Proof of (8.5) for free waves, log loss.* As before, it suffices to estimate

$$\left| \int H^{-1}(\bar{\phi}_{R_1} \partial_1 \phi_{R_2}) \bar{\phi}_{R_3} \partial_2 \phi_{R_4} \, dxdt - \int H^{-1}(\bar{\phi}_{R_1} \partial_2 \phi_{R_2}) \bar{\phi}_{R_3} \partial_1 \phi_{R_4} \, dxdt \right| \quad (8.11)$$

where  $R_1, R_2$  are  $\alpha$ -separated at frequency  $\lambda$  and  $R_3, R_4$  are  $\alpha$ -separated at frequency  $\mu$ . We now split into two subcases according to the strength of the null form.

**Subcase I:** If there is  $\beta \gtrsim \alpha\lambda$  such that  $|\xi_2 \wedge \xi_4| \sim \beta$  for all  $\xi_2 \in R_2$  and  $\xi_4 \in R_4$  then we use the same argument as in Case 3c(i).

**Subcase II:** Suppose now that  $|\xi_2 \wedge \xi_4| \lesssim \alpha\lambda$  for all  $\xi_2 \in R_2$  and  $\xi_4 \in R_4$  and also that  $\mu \approx \lambda$ . Now we write the expression to control in the form

$$\alpha\lambda \left| \int P_\alpha H^{-\frac{1}{2}}(\bar{\phi}_{R_1} \phi_{R_2}) P_\alpha \bar{H}^{-\frac{1}{2}}(\bar{\phi}_{R_3} \phi_{R_4})(x, t) dt dx \right|$$

This is estimated by applying (6.9) to each of the two bilinear factors; there is a  $\log \lambda$  loss from the summation with respect to the  $H^{-1}$  modulation.

**Case 3c(ii), Step 2:** *Proof of (8.5) for  $U^2$  waves, log loss.* Here we will show that

$$|I_{3c}| \lesssim \log \lambda \|P_e \phi_\lambda^{(1)}\|_{|D|^{-\frac{1}{2}}U_e^2} \|P_e \phi_\lambda^{(2)}\|_{|D|^{-\frac{1}{2}}U_e^2} \|P_{\bar{e}} \phi_\mu^{(3)}\|_{|D|^{-\frac{1}{2}}U_{\bar{e}}^2} \|P_{\bar{e}} \phi_\mu^{(4)}\|_{|D|^{-\frac{1}{2}}U_{\bar{e}}^2}, \quad \mu \approx \lambda \quad (8.12)$$

As in the similar argument in Case 3c(i) the problem reduces to the case where each factor is localized in cubes of size  $\alpha \times \alpha\lambda$  located near the parabola. From here on, the proof proceeds exactly as in the free case, using the observation that the bilinear  $L^2$  bounds (6.8) and (6.9) extend to  $U_e^2$  functions. This is somewhat less obvious for (6.9); what helps is that the  $\alpha \times \alpha$  frequency localization allows for separation of variables in  $P_\alpha H^{-1}$ , reducing the problem to a purely temporal multiplier. But purely temporal multipliers interact well with the lateral  $U^2$  atomic structure.

**Case 3c(ii), Step 3:** *Proof of (8.10) for  $V^2$  waves, log loss.* Precisely, we will show that

$$|I_{3c}| \lesssim (\log \mu)^5 \|\phi_\lambda^{(1)}\|_{l^2 V^{2,\#}} \|\phi_\lambda^{(2)}\|_{l^2 V^{2,\#}} \|\phi_\mu^{(3)}\|_{l^2 V^{2,\#}} \|\phi_\mu^{(4)}\|_{l^2 V^{2,\#}}, \quad \mu \approx \lambda \quad (8.13)$$

The role of the  $l^2$  structure here is to allow localization to the unit time scale. Once this is done, from the fact that the  $U_\Delta^2$  and  $V_\Delta^2$  norms are equivalent at fixed modulation (see (4.10)), we have

$$\|P_e \phi_\lambda\|_{|D|^{-\frac{1}{2}}U_e^2} \lesssim \log \lambda \|\phi_\lambda\|_{V_\Delta^2}$$

Therefore (8.13) follows from (8.12).

## 9. QUINTIC TERMS

In this section, we focus on controlling the quintic terms  $N_{5,1}$ ,  $N_{5,2}$ , and  $N_{5,3}$  of the nonlinearity, defined respectively by (3.11), (3.12), and (3.13). By pairing with a wave  $\bar{\phi}$  and using duality as in the case of the trilinear terms, controlling these terms is equivalent to controlling the integral

$$I^6 = \int w_1 w_2 w_3 dx dt$$

with

$$w_1 = H^{-1}(\bar{\phi}_{\lambda_1}^{(1)} \partial \phi_{\lambda_2}^{(2)}), \quad w_2 = H^{-1}(\bar{\phi}_{\lambda_3}^{(3)} \partial \phi_{\lambda_4}^{(4)}), \quad w_3 = \phi_{\lambda_5}^{(5)} \bar{\phi}_{\lambda_6}^{(6)}$$

and its variations obtained by moving the derivative from one factor to the other in each pair and by replacing  $H$  by  $\bar{H}$ .

We denote by  $\lambda$  the largest of the  $\lambda_j$ 's (which must appear at least twice) and by  $\lambda_0$  the smallest. We also assume the normalization

$$\|\phi_{\lambda_j}^{(j)}\|_{l^2 V^{2,\#}} = 1$$

Then we need to establish the estimate

$$|I^6| \lesssim \lambda_0^s, \quad s > 0 \quad (9.1)$$

Adding frequency localizations to each of the bilinear expressions, we can replace  $w_1, w_2, w_3$  by

$$w_1 = P_{\mu_1} H^{-1}(\bar{\phi}_{\lambda_1}^{(1)} \partial \phi_{\lambda_2}^{(2)}), \quad w_2 = P_{\mu_2} H^{-1}(\bar{\phi}_{\lambda_3}^{(3)} \partial \phi_{\lambda_4}^{(4)}), \quad w_3 = P_{\mu_3}(\phi_{\lambda_5}^{(5)} \bar{\phi}_{\lambda_6}^{(6)}) \quad (9.2)$$

The  $\mu_j$  summation is straightforward since we must have either  $\mu_j = \lambda$  or  $\mu_j \leq \lambda_0$ . We can harmlessly order the frequencies as follows:

$$\lambda_1 \leq \lambda_2, \quad \lambda_3 \leq \lambda_4, \quad \lambda_5 \leq \lambda_6$$

Each of the three bilinear expressions is called unbalanced if  $\mu_j \approx \lambda_{2j}$  and balanced otherwise (i.e. if  $\mu_j \ll \lambda_{2j-1} \approx \lambda_{2j}$ ). We begin by dispensing with the easy cases:

**Case 1: Unbalanced-unbalanced-either** Lemma 7.2, the Sobolev embedding (7.7), and the  $L^4$  Strichartz estimate (4.2) allow us to place each of the first two bilinear factors in  $L^4$  and the third one in  $L^2$ .

**Case 2: Unbalanced-balanced-either** Here we put the term identified as balanced in  $L_e^{\infty,3}$ , the one identified as unbalanced in  $L_e^{2,6}$ , and the remaining  $\phi^2$  term in  $L^2$ . This we can achieve thanks to Lemma 7.2, the Sobolev embedding (7.8), and the  $(q, r) = (4, 4)$  Strichartz estimate (4.2).

**Case 3: Balanced-balanced-either** This case is also easy if high modulations are present in one of the balanced couples, say the first one. Indeed, if one of the modulations there is much larger than  $\mu_1 \lambda_1$ , then we can use estimate (7.4) to conclude as in Case 2. Thus we are left with the low modulation case, where after some relabeling we need to consider  $w_1, w_2, w_3$  of the form

$$\begin{aligned} w_1 &= P_{\mu_1} H^{-1}(\overline{Q_{\lesssim \mu_1 \lambda_1} \phi_{\lambda_1}} \partial Q_{\lesssim \mu_1 \lambda_2} \phi_{\lambda_1}), \\ w_2 &= P_{\mu_2} H^{-1}(\overline{Q_{\lesssim \mu_2 \lambda_2} \phi_{\lambda_2}} \partial Q_{\lesssim \mu_2 \lambda_2} \phi_{\lambda_2}) \\ w_3 &= P_{\mu_3}(\phi_{\lambda_3} \bar{\phi}_{\lambda_4}) \end{aligned}$$

By the Littlewood-Paley trichotomy, the three output frequencies  $\mu_1, \mu_2$  and  $\mu_3$  are not independent. We denote the larger magnitude scale, which is shared by two frequencies, by  $\mu_{\text{hi}}$ , and the smaller one by  $\mu_{\text{lo}}$ .

We begin with a simpler computation, which applies under the assumption that  $\lambda_1 \notin \{\lambda_3, \lambda_4\}$ . Under this condition we claim that

$$\|w_1 w_3\|_{L^1} \lesssim \log^2(\min\{\lambda_1, \lambda_4\}) \frac{\lambda_1}{\mu_1(\lambda_1 + \lambda_3)^{\frac{1}{2}}(\lambda_1 + \lambda_4)^{\frac{1}{2}}} \quad (9.3)$$

To see this we begin by harmlessly inserting angular localizations  $P_e$  in the two factors in  $w_1$ . Making the stronger assumption that  $P_e Q_{\lesssim \mu_1 \lambda_1} \phi_{\lambda_1}$  is in either  $U_\Delta^2$  or  $U_e^2$  we can localize both factors further to nearby frequency cubes  $R_1, R_2$  of size  $\mu \times \mu \times \lambda\mu$  using the square summability provided by Lemma 4.11. Using the kernels  $K_{\mu_1}$  and  $K_{\mu_3}^0$  of  $P_{\mu_1} H^{-1}$ , respectively  $P_{\mu_3}$ , we can write

$$\begin{aligned} w_1 w_3(t, x) &= \int K_{\mu_1}(s, y) (\bar{P}_{R_1} Q_{\lesssim \mu_1 \lambda_1} \phi_{\lambda_1} \partial P_{R_2} Q_{\lesssim \mu_1 \lambda_2} \phi_{\lambda_1})(t - s, x - y) \\ &\quad K_{\mu_3}^0(y_1) (\phi_{\lambda_3} \bar{\phi}_{\lambda_4})(t, x - y_1) ds dy dy_1 \end{aligned}$$

Then we match one  $\phi_{\lambda_1}$  factor with  $\phi_{\lambda_3}$  and one with  $\phi_{\lambda_4}$  and apply twice the estimate (6.16), also noting that  $\|K_{\mu_1}\|_{L^1} \lesssim \mu_1^{-2}$  and  $\|K_{\mu_3}^0\|_{L^1} \lesssim 1$ . This gives the bound (9.3) but using  $U^2$  type norms for the  $\phi_{\lambda_1}$  factors. The transition to  $V^2$  norms is made trivially if  $\lambda_1 \lesssim \lambda_4$  (which allows  $\log \lambda_1$  losses). Else we use the  $U_e^2$  norms and transition to  $V_e^2$  norms via Lemma 4.8.

Then by using (9.3) for  $w_1 w_3$  and the  $L^\infty$  bound (6.2) for  $w_2$  we obtain

$$|I^6| \lesssim (\log \lambda_0)^2 \lambda_1 \lambda_2 \mu_1^{-2} \frac{\mu_1}{(\lambda_1 + \lambda_3)^{\frac{1}{2}} (\lambda_1 + \lambda_4)^{\frac{1}{2}}} \frac{\mu_2}{\lambda_2} = (\log \lambda_0)^2 \frac{\lambda_1}{(\lambda_1 + \lambda_3)^{\frac{1}{2}} (\lambda_1 + \lambda_4)^{\frac{1}{2}}} \frac{\mu_2}{\mu_1} \quad (9.4)$$

This will be used to dispense with some of the easier cases in the sequel.

**Case 3(a): Balanced-balanced-unbalanced.** This case is fully handled via (9.4). To see that we note that in this case we have  $\mu_3 = \lambda_3$ .

If  $\mu_3 = \mu_{lo}$  then we must have  $\mu_1 = \mu_2 \geq \lambda_3$  which implies that  $\lambda_1, \lambda_2 \gg \lambda_3$ . Thus (9.4) applies and suffices.

If  $\mu_3 = \mu_{hi}$  then we can assume that  $\mu_1 = \mu_3 \gg \mu_2$ . Hence  $\lambda_1 \gg \lambda_3$ , and (9.4) again applies and suffices.

**Case 3(b): Balanced-balanced-balanced.** Here we have  $\lambda_4 = \lambda_3 \gg \mu_3$ . We first use (9.4) to reduce the number of cases.

If  $\lambda_1 \neq \lambda_3$  and  $\lambda_2 \neq \lambda_3$  then we can assume that  $\mu_2 \leq \mu_1$  and (9.4) is enough.

If  $\lambda_1 \neq \lambda_3$  but  $\lambda_2 = \lambda_3$  then (9.4) suffices only if  $\mu_1 \gtrsim \mu_2$ .

Thus we are left with two cases:

- $\lambda_1 \neq \lambda_2 = \lambda_3$  and  $\mu_1 \ll \mu_2 = \mu_3 \ll \lambda_3$ .
- $\lambda_1 = \lambda_2 = \lambda_3$  and  $\mu_1 \leq \mu_2$ .

At this point the factors in  $w_1$  and  $w_2$  are restricted to low modulations, but not those in  $w_3$ . However it is easy to reduce the problem in both of these cases to small modulations ( $\leq \mu_3 \lambda_3$ ). Indeed, suppose that one of the  $\phi_{\lambda_3}$  has high modulation. Then we use (6.2) to bound  $w_1$  in  $\mu_1 L^\infty$ .

For  $w_2$  we use (6.17) and Bernstein to bound it in  $\mu_2^{-1} \lambda_2^{\frac{1}{2}} L^2 L^4$  while  $w_3$  is in  $\lambda_3^{-\frac{1}{2}} L^2 L^{-\frac{4}{3}}$ , again by Bernstein. Thus from here on we assume that

$$w_3 = P_{\mu_3}(\overline{Q_{\lesssim \mu_3 \lambda_3} \phi_{\lambda_3}} Q_{\lesssim \mu_3 \lambda_3} \phi_{\lambda_3})$$

**Case 3(b)(i):**  $\lambda_1 \neq \lambda_2 = \lambda_3$  and  $\mu_1 \ll \mu_2 = \mu_3 \ll \lambda_3$ . Even though (9.4) does not cover this case in full, we can still get some use out of it if we restrict ourselves to high modulations of  $w_1$ , namely

$$w_1^{hi} = P_{\mu_1} H^{-1} Q_{\gtrsim \mu_2^2}^0(\overline{Q_{\lesssim \mu_1 \lambda_1} \phi_{\lambda_1}} \partial Q_{\lesssim \mu_1 \lambda_2} \phi_{\lambda_1})$$

Then the  $L^1$  norm of the kernel for  $P_{\mu_1} H^{-1} Q_{\gtrsim \mu_2^2}^0$  is  $\mu_2^{-2}$ , which suffices.

It remains to consider the low modulations of  $w_1$ ,

$$w_1^{lo} = P_{\mu_1} H^{-1} Q_{\ll \mu_2^2}^0(\overline{Q_{\lesssim \mu_1 \lambda_1} \phi_{\lambda_1}} \partial Q_{\lesssim \mu_1 \lambda_2} \phi_{\lambda_1})$$

But in this case the modulations of  $w_2$  and  $w_3$  are comparable, say equal to  $\nu > \mu_2^2$  (or both  $\leq \nu = \mu_2^2$ ). Then we apply the  $L^\infty$  bound (6.2) to  $w_1^{lo}$  and the  $L^2$  bound (6.17) to  $Q_\nu^0 w_2$  and  $Q_\nu^0 w_3$  to obtain

$$|I^6| \lesssim \frac{\lambda_1 \lambda_2}{\nu} \frac{\mu_1}{\lambda_1} \frac{\nu}{\mu_2 \lambda_2} \lesssim 1$$

**Case 3(b)(ii):**  $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$ .

In this case we introduce full modulation truncations for each of the three bilinear factors and consider

$$I^6 = \int Q_{\nu_1}^0 w_1 Q_{\nu_2}^0 w_2 Q_{\nu_3}^0 w_3 \, dx dt$$

where

$$\mu_1^2 \leq \nu_1 \leq \mu_1 \lambda_1, \quad \mu_2^2 \leq \nu_2 \leq \mu_2 \lambda_2, \quad \mu_3^2 \leq \nu_3 \leq \mu_3 \lambda_3 \quad (9.5)$$

The reason for the lower bounds is that the symbol of  $H^{-1}$  no longer changes at lower modulations. Implicitly we allow a slight abuse of notation, where for  $\nu_i = \mu_i^2$  we replace  $Q_{\nu_i}^0$  by  $Q_{\lesssim \mu_i^2}^0$ . The two largest modulations are comparable; we use  $\nu_{hi}$  to denote their scale, along with  $\nu_{lo}$  to denote the scale of the remaining modulation,  $\nu_{lo} \lesssim \nu_{hi}$ .

Finally, we remark that the modulation summation only yields acceptable logarithmic losses; in particular the  $U^2$  and  $V^2$  norms are logarithmically close.

**Subcase I.**  $\mu_{lo}$  is paired with  $\nu_{lo}$ . Then we apply the bilinear  $L^\infty$  estimate (6.2) for the corresponding factor and the bilinear  $L^2$  bound (6.17) for the remaining factors to obtain

$$|I^6| \lesssim (\log \lambda_0)^4 \frac{\lambda^2}{\nu_1 \nu_2} \frac{\nu_{lo} \mu_{lo}}{\lambda} \frac{\nu_{hi}}{\mu_{hi} \lambda} \lesssim (\log \lambda_0)^4 \frac{\mu_{lo}}{\mu_{hi}}$$

**Subcase II.**  $\nu_3 = \nu_{lo}$ ,  $\mu_1 = \mu_{lo}$ . We apply the bilinear  $L^\infty$  estimate (6.2) for  $P_{\nu_1} w_1$  and the  $L^2$  bound (6.17) for the remaining factors and conclude as above.

**Subcase III.**  $\nu_1 = \nu_{lo}$ ,  $\mu_1 = \mu_{hi}$ . Suppose that  $\mu_2 = \mu_{lo}$ , as the argument is similar in the other case. This is the most difficult case. We begin with several reductions.

We first localize each pair of factors to small angles using multipliers  $P_{\mathbf{e}_1}$ ,  $P_{\mathbf{e}_2}$  and  $P_{\mathbf{e}_3}$ . Since  $\log \lambda$  losses are allowed in this case, it suffices to work with factors in the spaces  $X^{0, \frac{1}{2}, 1}$ . Further, by orthogonality we can reduce the problem to the case when both  $\phi_{\lambda_j}$  factors are frequency localized to cubes of size  $\mu_j \times \mu_j \times \nu_j$ . Then  $w_j$  has a similar localization, and the  $L^2$  bound given by (6.17).

If both  $\phi_{\lambda_j}$  were free waves, then in effect  $w_j$  would be localized in smaller regions, namely a tilted cube  $R_j$  of size  $\mu_j \times \frac{\mu_j^2}{\lambda} \times \nu_j$ . Then we can estimate

$$\begin{aligned} \|w_1 w_2\|_{L^2} &\lesssim \|w_1\|_{L^2} \|w_2\|_{L^2} \sup_{(\tau, \xi)} |R_1 \cap (\tau, \xi) - R_2| \\ &\lesssim \frac{\lambda^2}{\nu_{lo} \nu_{hi}} \left( \frac{\nu_{lo}}{\mu_{hi} \lambda} \right)^{\frac{1}{2}} \left( \frac{\nu_{hi}}{\mu_{lo} \lambda} \right)^{\frac{1}{2}} \left( \nu_{lo} \mu_{lo} \frac{\mu_{lo}^2}{\lambda} \right)^{\frac{1}{2}} \\ &\lesssim \left( \frac{\lambda \mu_{hi}}{\nu_{hi}} \right)^{\frac{1}{2}} \left( \frac{\mu_{lo}}{\mu_{hi}} \right) \end{aligned}$$

Combined with the  $L^2$  bound for  $w_3$ , this leads to the desired conclusion.

In order to gain a similar localization in the case when the factors are  $X^{0, \frac{1}{2}, 1}$  functions, we foliate them with respect to the modulation, with integrability with respect to the modulation parameter. For pointwise fixed modulation parameters in all six factors the above argument still applies, and the conclusion follows.

## 10. CONCLUSION

It remains to show that the error terms may be controlled. Recall from §3 that we have

$$\begin{aligned}
E_1 &= H^{-1}(H^{-1}A_x(0)|\phi|^2)\partial\phi \\
E_2 &= H^{-1}\partial(H^{-1}A_x(0)|\phi|^2)\phi \\
E_3 &= H^{-1}A_x(0)H^{-1}(\bar{\phi}\partial\phi)\phi \\
E_4 &= (H^{-1}A_x(0))^2\phi + H^{-1}A_x(0)B\phi + B^2\phi \\
E_5 &= H^{-1}(B|\phi|^2)\partial\phi \\
E_6 &= H^{-1}\partial(B|\phi|^2)\phi \\
E_7 &= H^{-1}(\bar{\phi}\partial\phi)B\phi
\end{aligned}$$

Some of these terms are related via duality. In particular, estimating  $E_1$  paired with  $\phi$  is equivalent to estimating

$$H^{-1}A_x(0)|\phi|^2H^{-1}(\phi\partial\phi)$$

In fact control on this term also gives control on  $E_2$  paired with  $\phi$ . Similarly, estimating  $E_5$  paired with  $\phi$  is equivalent to estimating

$$B|\phi|^2H^{-1}(\phi\partial\phi)$$

Control on this term also gives control on  $E_6$  paired with  $\phi$ .

To obtain estimates, we use (7.1) for  $H^{-1}A(0)$ , which gives  $H^{-1}A(0) \in L_{t,x}^4[0,1] \cap L_{\mathbf{e}}^{2,6}[0,1]$ ; (7.5) for  $B = H^{-1}(A|\phi|^2)$ , which provides  $B \in H^{-\frac{1}{2}}L_{t,x}^2[0,1] \in L_{t,x}^4[0,1] \cap L_{\mathbf{e}}^{2,6}[0,1]$ ; Strichartz for  $\phi$ , providing  $\phi \in L_{t,x}^4$ ; and finally the  $L_{\mathbf{e}}^{\infty,3}$  or  $H^{-\frac{1}{2}}L_{t,x}^2$  bounds on  $H^{-1}(\phi\partial\phi)$  coming from Lemma 7.2.

In fact, the estimates on  $H^{-1}A(0)$ ,  $B$ ,  $\phi$ , and  $H^{-1}(\phi\partial\phi)$  come with extra regularity. This extra regularity guarantees all of the frequency summations.

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